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A CRITERION FOR THE DEVELOPMENT OF THE FULL PLASTIC MOMENT AT ONE END OF A UNIFORM STRUT PRIOR TO INSTABILITY

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CONTENTS

Plastic design methods for steel-framed structures involve an assumption that members do not become unstable prior to the development of sufficient hinges to form a mechanism. Some check on this assumption is desirable and a basic problem is that of a member subjected to combined axial stress and unequal major-axis bending moments at its ends; the question is then whether the full plastic moment can be developed at one end without prior instability. In previous theories, Horne has tackled this problem indirectly by considering an 'equivalent' problem in which the major-axis bending moment is uniform along the member. The actual loading problem is now analysed directly and a criterion of critical slenderness ratio is developed which can be linked to physically plausible causes of instability. The theory is in reasonable agreement with relevant published experimental data except in one instance; but the data are very limited and many more are needed before any theory can be verified or disproved with confidence.

Notation

a_n	coefficient in eigenfunction expansion of u_0
a_y	distance of extreme fibre from the neutral axis for minor-axis bending
A', B'	constants
A	cross-sectional area of strut
b, b'	defined by equations (A 6) and (A 11) respectively
B_1	major flexural rigidity
B_2	minor flexural rigidity
B_y	value of B_2 for fully elastic section
C	St Venant torsional rigidity
C_0	value of C for fully elastic section
E	Young's modulus
f_L	yield stress
$f(z)$	defined by equation (6)
F	function of β tabulated by Horne (1956)
F_n	$F_n(P, M^2) = 0$ is eigenfunction relation for the nth buckling mode of a straight equivalent elastic strut
H_1	function of ζ allowing for the effect of yielding on the lowest Euler critical load
H_2	function of ζ and β allowing for the effect of yielding on the critical moment for lateral instability of an equivalent elastic strut under end moments M and βM
I_1, I_2, I_3, I_4	defined by equation (C 1)
J_0, J_1	Bessel functions
k_n	defined by equation (14)
K	defined by equation (30)
l	length of strut
L_1, L_2, L_3, L_4	defined by equation (A 29)
m	index in assumed forms for ϕ, equations (A 19) and (A 32)
M	applied major-axis bending moment at the top of the strut
M_ξ	major-axis bending moment at general section of the strut
M_η	minor-axis bending moment at general section of the strut
M_p	fully plastic moment for major-axis bending
M_L	moment at first yield for major-axis bending
M_0	critical moment for lateral instability of a uniform elastic strut subjected to end moments M and βM
M_1	defined by equation (C 6)
M_{10}	critical moment for lateral instability of a non-uniform elastic strut under end moments M and βM
$n = p/f_L$	
p	axial stress
P	axial load
P_0	Euler critical load for minor-axis instability of uniform elastic strut simply supported at its ends
P_1	defined by equation (C 6)

P_{10} Euler critical load for minor-axis instability of non-uniform elastic strut
r_y radius of gyration of section about the minor axis
$S(n) = M_L/M_p$
u deflexion in direction of major axis
u_0 initial value of u prior to any loading
$u_n(z)$ variation of u with z in nth buckling mode
v deflexion in the direction of the minor axis
z distance along the centre line of a straight strut measured from the bottom end
$z_1 = z/l$
Z_x elastic major-axis section modulus
Z_y elastic minor-axis section modulus
X, X_n end shear reaction for case of foot clamped about minor axis
α defined by equation (37)
β ratio of applied end moments
γ_y defined by equation (33)
ϕ twisting rotation of section
ϕ_n twisting rotation of section in nth buckling mode
$\psi(z_1)$ defined by equation (29)
ψ_m maximum value of ψ for varying z_1
$\chi(z_1)$ defined by equation (24)
μ defined by equation (48)
ζ value of z_1 for section that is just beginning to yield when $M = M_p$ at top end of strut
ζ_0 defined by equation (40)
ζ_1 defined by equation (45)
ζ_m value of ζ at maximum point of curve DEF, figure 4
λ defined by equation (A 20)

1. Introduction

In the plastic method of design for steel-framed structures it is assumed that the development of plastic hinges allows redistribution of moments under increasing load until sufficient hinges occur to cause collapse of the structure as a mechanism. The method involves an assumption that a member does not collapse by instability prior to the formation of a plastic hinge in it, and some check on this assumption is desirable when using the plastic design method. Horne (1956) has derived theoretical curves for checking this assumption in certain cases which are common in practice. The problem which he considers is that of a uniform strut with both ends completely restrained against twisting and both ends simply-supported about the major and minor axes; the strut is subjected to an axial stress p together with end moments about the major axis, M at one end and βM at the other end, where $-1 \leqslant \beta \leqslant 1$; the extreme case $\beta = 1$ corresponds to uniform bending about the major axis, while the other extreme case $\beta = -1$ corresponds to bending about the major axis with equal double curvature. Then, for a given strut section and given p, β, Horne's final curves determine the limiting slenderness-ratio l/r_y such that the fully plastic moment $M = M_p$ can be developed without instability for all smaller slenderness ratios. In this problem, instability prior to $M = M_p$ in a test with monotonically increasing loads will correspond to the attainment of a state of maximum loading, followed by sudden collapse under dead loading or followed by increasing deformation under decreasing load if the loading is strain controlled.

Horne assumes in his analysis that the behaviour of the strut under the loading p, M and βM, is the same as that of the strut subjected to an axial stress p and a uniform major-axis bending moment $M/\surd F$, where F is a function of β determined by the condition that when $p = 0$ and the strut remains elastic, the critical moment is the same for the two types of loading. His criterion for the limiting slenderness ratio is then that the yield stress is just attained at mid-span for this equivalent uniform moment problem. The chief objection to these assumptions is that no clear physical picture can then be obtained of what Horne's criterion means in terms of the behaviour of the strut under the actual loading by p and end moments M and βM. Also, his assumptions lead to the conclusion that there is a whole range of loading combinations of p and β, such that no strut, however short, can develop the full plastic moment; this range corresponds to combinations of axial stress p and uniform moment $M/\surd F$ which would cause yield for $M < M_p$ even for a perfectly straight strut. The exclusion of such cases errs on the safe side but can be unduly restrictive in practice. Recently, Horne (1964 *a*, *b*) has presented a modified theory which, in particular, avoids the second of the preceding objections by including an overriding criterion which implies that sufficiently short struts can develop the full plastic moment for all combinations of β and p. But the modified main theory is still based on the equivalent moment case as in the earlier theory, and it is therefore still open to the preceding objection that it has no clear interpretation in terms of the physics of the actual loading case. The present analysis has been developed to overcome both the preceding objections to Horne's earlier theory. It refers throughout to the actual loading by an axial stress p and end moments M and βM, and a criterion of critical slenderness ratio is linked to physically plausible causes of instability.

Before commencing the analysis, certain general assumptions will be stated. First, in so far as the deformation of the strut depends on the loading, it will be assumed that at any stage of loading the deformation depends only on the values of p, M and βM and not on the particular manner or order in which these loads are applied. This seems a reasonable assumption provided all loads have been increased monotonically from zero up to this stage. On this basis, it will suffice to consider a particular order of loading and, for all further discussion, it will be assumed that the final axial load is first applied, followed by end moments M and βM, with β constant and M increasing steadily from zero. For brevity, this procedure will be referred to as *standard* loading, and the end subjected to the larger moment M will be called the *top* end.

For standard loading on struts of the same section which collapse before $M = M_p$, it will be assumed that the collapse value of M will decrease monotonically as the strut length increases. It then follows that there will be a critical length such that all shorter lengths will develop the full plastic moment, while all longer lengths will collapse prior to $M = M_p$. The determination of this critical length, or equally the equivalent critical slenderness-ratio, is the ultimate objective of the analysis.

Warping rigidity will be neglected; this is expected to introduce an error on the safe side. The Wagner effect will also be neglected; this errs on the unsafe side but the error is likely to be small and less than the safe error due to neglect of warping rigidity over the range of the later numerical calculations.

The effect of yield on lateral stability will be considered in terms of a related reduction in the minor flexural rigidity B_2 as discussed quantitatively in a later section. Here we note only that the equations of equilibrium for the yielding strut at a given stage of yielding will then be the same as those for an elastic strut of non-uniform rigidity. Such a strut, differing from the actual strut in that its rigidities will be assumed constant independent of load, will be termed an *equivalent elastic strut*. The deformation of the equivalent elastic strut will coincide with that of the actual strut at the particular stage of loading, and an infinite set of equivalent elastic struts is postulated to cover the change of rigidity with yielding of the actual strut. In order to derive the basic equilibrium relation for the yielding strut, we proceed by first considering generally the equilibrium equations for an elastic strut of given non-uniform rigidity.

2. The case of a strut simply supported about the minor axis at its ends

2·1. *Equilibrium conditions*

We consider an equivalent elastic strut for which the flexural rigidities B_1, B_2 and the torsional rigidity C are assumed to be known functions of the distance z along the strut and are independent of the loading.

Both ends of the strut are assumed to be located in position, fully restrained against twisting and simply supported about the minor axis. The initial imperfection u_0 is also assumed to satisfy simply-supported end conditions. The relevant equations are

$$\left.\begin{aligned} u = u_0 = 0, \\ B_2\frac{d^2u}{dz^2} = B_2\frac{d^2u_0}{dz^2} = 0 \end{aligned}\right\} z = 0 \quad \text{and} \quad z = l, \tag{1}$$

$$\phi = 0, \quad z = 0 \quad \text{and} \quad z = l. \tag{2}$$

The strut is loaded by an axial force P and by major-axis end moments, M at the top and βM at the bottom, where $-1 \leqslant \beta < 1$. We shall assume that the major flexural rigidity B_1 is large compared to the minor flexural rigidity B_2 so that, in particular, we shall neglect the term Pv in the bending moment about the major axis. The equations of equilibrium are easily derived from statics and geometry in a manner analogous to that used, for example, by Timoshenko & Gere (1961) for the simpler problem of uniform moment, and we find

$$B_1\frac{d^2v}{dz^2} = -Mf(z), \tag{3}$$

$$B_2\left(\frac{d^2u}{dz^2} - \frac{d^2u_0}{dz^2}\right) + Pu = -\phi Mf(z), \tag{4}$$

$$\frac{d}{dz}\left(C\frac{d\phi}{dz}\right) = Mf(z)\frac{d^2u}{dz^2}, \tag{5}$$

where
$$f(z) = \frac{z}{l} + \beta\left(1 - \frac{z}{l}\right). \tag{6}$$

Equations (4) and (5) together with the end conditions (1) and (2) suffice to specify completely the analytical problem of determining u and ϕ when B_2, C and u_0 are known functions of z. Further, we can eliminate ϕ from (4) and (5) to obtain the following equation for u

$$\frac{d}{dz} C \frac{d}{dz}\left[\frac{B_2}{f(z)}\left(\frac{d^2u}{dz^2}-\frac{d^2u_0}{dz^2}\right)+\frac{Pu}{f(z)}\right]+M^2 f(z)\frac{d^2u}{dz^2}=0, \tag{7}$$

which is of the fourth order in u, with the four corresponding end conditions given in (1).

We proceed by considering first the special case of an initially straight strut, $u_0 \equiv 0$, for which we know the problem to be of eigenvalue type with an infinite set of buckling modes. For any typical mode we write $u = u_n(z)$, $\phi = \phi_n(z)$, $u_0 = 0$ whence from equation (7) we obtain

$$\frac{d}{dz} C \frac{d}{dz}\left[\frac{B_2}{f(z)}\frac{d^2u_n}{dz^2}+\frac{Pu_n}{f(z)}\right]+M^2 f(z)\frac{d^2u_n}{dz^2}=0, \tag{8}$$

which has associated end conditions

$$u_n = B_2\, d^2u_n/dz^2 = 0, \quad z = 0 \quad \text{and} \quad z = l, \tag{9}$$

while from equation (4) we find

$$B_2(d^2u_n/dz^2) + Pu_n = -\phi_n M f(z). \tag{10}$$

Now, for a given value of β, the typical buckling mode will correspond to a specific eigen-relation between P and M^2 which can be written generally in the form

$$\left.\begin{aligned} F_n(P, M^2) &= 0, \\ \text{when} \qquad u &= u_n. \end{aligned}\right\} \tag{11}$$

There will be an infinite number of such relations and if they were plotted with P as ordinate and M^2 as abscissa, we expect a series of curves of the qualitative form shown in figure 1 for $P \geqslant 0$. It should be noted that the eigenfunctions (u_n, ϕ_n) corresponding to points on the curve $F_n = 0$ will in general vary for differing points on this curve. For later use, we shall consider the set of eigenfunctions $u_1, u_2, \ldots$ and associated $\phi_1, \phi_2, \ldots$ corresponding to the points $Q_1, Q_2, \ldots$ on a radius vector in figure 1, for which the ratio P/M^2 is the same; it can be shown that these eigenfunctions satisfy the orthogonal relation

$$\int_0^l B_2 \frac{d^2u_n}{dz^2}\frac{d^2u_m}{dz^2}\, dz = 0 \quad (m \neq n). \tag{12}$$

We return now to the more general case where the equivalent elastic strut has a given initial imperfection u_0. The basic equations for u are then (7) and (1), and we assume for given β that a given state of loading (P, M^2) has been reached prior to instability, this state being represented by the point Q in figure 1.

The solution of (7) can be examined by expanding the known initial imperfection in a series of the form

$$u_0(z) = \sum_{n=1}^{\infty} a_n u_n(z) \tag{13}$$

in terms of the eigenfunctions $u_1, u_2, \ldots$ which are associated with the points $Q_1, Q_2, \ldots$ where the radius vector through Q (figure 1) cuts the curves $F_n = 0$.

The coefficients a_n in equation (13) could be determined by using the orthogonal relation (12) in the usual manner. But we shall not make use of this process since initial imperfections are never known with precision in practice and it is simpler to make direct assumptions for the magnitudes of the coefficients a_n.

To solve equation (7), we now expand u in the series

$$u = \sum_{n=1}^{\infty} k_n a_n u_n(z), \tag{14}$$

where the coefficient multipliers k_n are at present unknown, and we note that (14) satisfies the end conditions (1), since they are satisfied separately by each $u_n(z)$.

We now substitute (13) and (14) in (7) and since the equations are linear, we can seek a solution by making the typical nth terms satisfy (7), whence we derive the equation

$$\frac{d}{dz} C \frac{d}{dz}\left[\frac{B_2}{f(z)}\frac{d^2u_n}{dz^2} + \left(\frac{k_n P}{k_n-1}\right)\frac{u_n}{f(z)}\right] + \left(\frac{k_n M^2}{k_n-1}\right) f(z)\frac{d^2u_n}{dz^2} = 0, \tag{15}$$

after dividing through by $(k_n-1)\,a_n$.

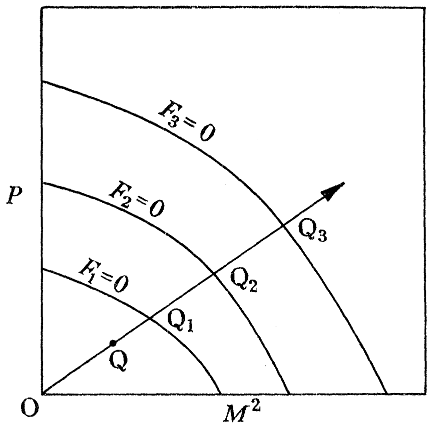

FIGURE 1. Qualitative shape of eigenrelations.

The unknown quantity in equation (15) is k_n and not u_n, since u_n has already been defined as the eigenfunction corresponding to the point Q_n where the radius vector through Q (figure 1) cuts the curve $F_n = 0$. Thus u_n must satisfy the special case of equation (8) in which (M^2, P) are identified with the coordinates of Q_n. Now equation (15) differs only from equation (8) in that $k_n P/(k_n-1)$ replaces P, and $k_n M^2/(k_n-1)$ replaces M^2. It follows that equation (15) will be satisfied if k_n can be chosen so that the point of coordinates $\{k_n M^2/(k_n-1),\, k_n P/(k_n-1)\}$ in figure 1 is identified with the point Q_n, noting that (M^2, P) are the coordinates of the point Q. But the ratio of the coordinates of the point $\{k_n M^2/(k_n-1),\, k_n P/(k_n-1)\}$ is M^2/P so that this point lies on the radius vector through Q, independent of the value of k_n; hence this point will coincide with Q_n, and equation (15) will be satisfied, if we make it lie also on the curve $F_n = 0$, which gives the equation

$$F_n\left[\frac{k_n P}{k_n-1}, \frac{k_n M^2}{k_n-1}\right] = 0 \tag{16}$$

from which k_n can be determined for any given (M^2, P), assuming the function F_n to be known.

It is interesting to note that equation (16) has a simple geometrical interpretation in figure 1. It means that if both the coordinates of Q are increased by the same factor

$k_1/(k_1-1)$, the resulting point will lie on the curve $F_1 = 0$ and hence will correspond to the point Q_1 in figure 1. Thus

$$\left.\begin{aligned} \frac{OQ_1}{OQ} &= \frac{k_1}{k_1-1}, \\ k_1 &= \frac{OQ_1}{QQ_1}. \end{aligned}\right\} \tag{17}$$

whence

Similarly, $k_2 = OQ_2/QQ_2$ and so on for higher values of n.

In practice, instability will always occur in the lowest mode and the conditions immediately prior to collapse will correspond to QQ_1 small, with k_1 large while $k_2, k_3, \ldots$ are of order unity. Also, in practice, the initial imperfection will usually involve a fundamental mode component of at least the same order of magnitude as the higher components in (13). Hence near collapse where k_1 is large, it is in general a reasonable approximation to assume that the lowest mode is dominant and to neglect the higher modes. Adopting this approximation, the basic results for further analysis are

$$\left.\begin{aligned} u = k_1 u_0 &= k_1 a_1 u_1(z), \\ F_1\left[\frac{k_1 P}{k_1-1}, \frac{k_1 M^2}{k_1-1}\right] &= 0, \end{aligned}\right\} \tag{18}$$

where u_1 and F_1 correspond to the lowest buckling mode of the straight equivalent elastic strut.

To make further progress, we need to specify the form of the function F_1 but the exact form, even if it could be found, is likely to be much too complex for use in the subsequent analysis. Instead we shall use the approximate form

$$\frac{P}{P_{10}} + \frac{M^2}{M_{10}^2} = 1, \tag{19}$$

where P_{10} is the lowest critical load when $M = 0$, while M_{10} is the lowest critical moment for $P = 0$. It is shown in appendix C that (19) is a safe approximation to the true relation $F_1 = 0$. Further, an exact solution obtained by Horne (1954) for the case of a uniform elastic strut indicates that (19) is quite a close approximation for $\beta \geqslant 0$ in this case.

Since the analysis is to be used for the yielding strut, it is convenient to express P_{10} and M_{10} in terms of their values P_0, M_0 for the actual strut while fully elastic prior to any yield. Then, since the effect of yielding will be to reduce the rigidity and hence the values of P_{10}, M_{10}, we allow for this by writing equation (19) in the form

$$\frac{P}{H_1 P_0} + \frac{M^2}{H_2 M_0^2} = 1, \tag{20}$$

where the factors H_1, H_2, are both $\leqslant 1$, and are considered in detail in appendix A.

Equation (20) is the assumed form for the relation $F_1 = 0$ and hence equation (18) for the initially curved strut leads to

$$\frac{k_1-1}{k_1} = \frac{P}{H_1 P_0} + \frac{M^2}{H_2 M_0^2} \tag{21}$$

as the expression for determining k_1. Here, P_0 is the usual Euler buckling load, while M_0

is the buckling moment when there is no axial load, both referring to the uniform elastic strut prior to any yield. Thus, if we now introduce Horne's symbol T, we can write

$$\left.\begin{aligned} C_0 &= \frac{Z_x^2 T}{A}, \\ P_0 &= \frac{\pi^2 B_y}{l^2} = \frac{\pi^2 E A r_y^2}{l^2}, \\ M_0^2 &= \frac{\pi^2 F(\beta)\, B_y C_0}{l^2} = \frac{\pi^2 F(\beta)\, E Z_x^2 T r_y^2}{l^2}, \end{aligned}\right\} \tag{22}$$

where values of the numerical factor F have been tabulated by Horne (1956).

The quantities H_1, H_2 in equation (21) allow for the effect of decreasing rigidity due to yielding, with $H_1 P_0$ equal to the buckling load of the equivalent elastic strut when $M = 0$, while $M_0\sqrt{H_2}$ is the buckling moment for this strut when $P = 0$. To determine H_1, H_2, we need to know the variation of B_2 and C with z and we therefore proceed to consider suitable assumptions for the effective rigidities of the strut section when it is partly plastic.

2·2. *The effect of yielding on torsional and flexural rigidities*

Neal (1950) has given some experimental evidence and a theoretical argument which suggest that the torsional rigidity of a strut section is unaffected, to the first order, by yielding under direct bending stresses. Hence we shall assume that the torsional rigidity remains constant over the whole length of the strut throughout yielding, so that

$$C = C_0, \tag{23}$$

where C_0 is the full elastic value of the torsional rigidity. Recently, Massey (1963) has produced contrary evidence to the effect that C is significantly reduced by yielding under direct stresses. However, close examination of this evidence indicates that it is based on relatively few experimental results and even if it is subsequently confirmed by further data, the error in the assumption (23) should be largely covered by the simultaneous neglect of warping rigidity in the present analysis. Hence it has not been considered worth while to revise the present analysis and calculations, which were largely completed prior to Massey's publication, in order to allow for a reduction of C after yielding.

Before considering the assumptions for the effect of yielding on the value of the minor flexural rigidity B_2, the possible causes of instability will be discussed. First, we note that the small shear stresses associated with torsion and shear force will only have a second-order effect on the maximum principal stress. This effect will be neglected so far as the extent of yielding at any section is concerned. This yielding will then depend on (i) the axial load P which is the same at all sections along the strut, (ii) the major-axis bending moment $M_\xi = Mf(z)$ which increases from the bottom to the top of the strut for $0 \leqslant \beta < 1$, and (iii) the minor-axis bending moment M_η which is zero at the two ends and reaches a maximum at some internal section. It is reasonably certain that the extent of yielding and the associated reduction in B_2 will increase as any one of P, M_ξ and M_η increases. Also for standard loading with P applied first, followed by the end moments with β constant, then the moment $M_\xi = Mf(z)$ will increase linearly with M, whereas M_η would be expected to increase rather faster than linearly as M increases. On these assumptions, struts can conveniently be divided broadly into three categories, namely:

(*a*) Struts for which M_η becomes sufficiently significant prior to yield so that the first yield occurs at a section away from the top end. In such cases, we should expect the section of maximum yielding and reduction in B_2 to move, if anything, further away from this end as M increases after first yield.

(*b*) Struts for which the first yield occurs at the top end but M_η becomes sufficiently significant before $M = M_p$ for the section of maximum yielding to move away from the top end while M is still less than M_p.

(*c*) Struts for which the first yield occurs at the top end and M_η remains small enough for the section of maximum yielding to remain at this end up to $M = M_p$, provided there is no prior instability.

In the present analysis, we shall neglect any strain-hardening after yield and follow Horne (1956) in assuming that B_2 is zero at any section which is fully plastic. Hence in categories (*a*) and (*b*) above, the fully plastic moment is tending to occur away from the ends and the strut is tending to become a mechanism for bending about the minor axis with zero lateral stability. Hence, instability must occur prior to $M = M_p$ and there is no point in analysing these first two categories in detail. Attention can therefore be concentrated on struts in the third category (*c*) above, subject to the introduction of some quantitative criterion to ensure that the first two categories are excluded. Here, there seems to be no relevant experimental data on the reduction in B_2 under the combined action of axial load and both major and minor axis-bending moments. In the absence of such data we shall assume that, for a given axial stress p, the rigidity B_2 decreases monotonically with increase in the quantity χ defined by

$$\left.\begin{aligned} \chi(z_1) &= \frac{|M_\xi|}{Z_x} + \frac{|M_\eta|}{Z_y}, \\ z_1 &= z/l \end{aligned}\right\} \qquad (24)$$

where

and the modulus signs denote as usual that both bending moments must be taken positively. The quantity χ is certainly the function governing the maximum stress in the elastic state and hence governing the question of when and where first yield occurs. Its use in relation to the reduction of B_2 after yielding is thought to be a safe assumption for the present purpose, since it probably overestimates the importance of M_η in relation to M_ξ when the section is partly plastic.

The numerical results from the present analysis will be concentrated on loading for which $\beta \geqslant 0$ and in these cases it is reasonably certain that the curve for minor-axis bending moment versus z is of single-curvature shape, assuming that the fundamental buckling mode is dominant. Thus in figure 2, the qualitative variation of χ with z will be of the form of the curve OAB or OCB, where the chord OB represents the linear contribution of the major-axis bending moment. The curve OAB then represents a case in which the maximum χ, and hence the maximum reduction in B_2 occurs at the top end, while the curve OCB shows this maximum occurring away from the top end. We require the condition that χ remains of the type OAB up to $M = M_p$, which is simply

$$\chi'(z_1) \geqslant 0, \quad z_1 = 1, \quad M = M_p. \qquad (25)$$

If this relation is satisfied, the section of maximum reduction in B_2 will occur at the top end for $M \leqslant M_p$; it remains to consider the condition for no instability before $M = M_p$ for a strut in this category. For this purpose the following assumptions will be made:

(i) When the fully plastic moment $M = M_p$ is developed at the top end, some yielding will have occurred throughout the upper portion of the strut, extending from the top $z_1 = 1$ to the section $z_1 = \zeta$ (say), while the lower portion $0 \leqslant z_1 \leqslant \zeta$ remains wholly elastic.

(ii) The section $z_1 = \zeta$ is determined by the condition that it can be regarded as just wholly elastic with a maximum longitudinal stress, due to the combined axial and bending stresses, just equal to the yield stress f_L.

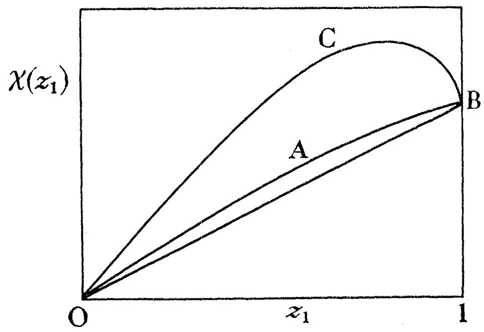

FIGURE 2. Qualitative shapes for $\chi(z_1)$.

(iii) When $M = M_p$, the flexural rigidity B_2 varies linearly with distance along the strut in the yielded portion, from zero value at the top end to the full elastic value B_y at $z_1 = \zeta$, so that

$$\left.\begin{aligned} B_2 &= B_y \qquad\qquad (0 \leqslant z_1 \leqslant \zeta), \\ B_2 &= B_y\left(\frac{1-z_1}{1-\zeta}\right) \quad (\zeta \leqslant z_1 \leqslant 1). \end{aligned}\right\} \tag{26}$$

The first assumption excludes the possibility that there are two or more distinct zones of yielding separated by an elastic portion of strut. This seems unlikely in cases where the full plastic moment is developed, though it may well occur for much slenderer struts which become unstable prior to $M = M_p$. The second assumption follows from the neglect of the second-order effect of shear stresses on yielding.

The third assumption, embodied in (26), is made as a simple adaptation of the corresponding assumption in Horne's analysis (1956) of the present problem. He states that it is a safe assumption, underestimating B_2 for a section under the action of a given axial load and increasing major-axis bending moment, to assume that B_2 decreases linearly as this moment increases above the yield moment and becomes zero when the section is fully plastic. Now the major-axis bending moment varies linearly along the strut and hence Horne's assumption would certainly lead to the form (26) if the stress due to the minor-axis bending moment were neglected in considering yield. But in the present analysis, we wish to take some account of the first order effect of this moment M_η and physically it would be more logical to assume some functional dependence of B_2 on M_η. However, this would undoubtedly lead to great complexity in the further analysis, which is scarcely warranted in the absence of experimental data on which to base this functional dependence. Hence for analytical simplicity, we have chosen the assumption (26) giving B_2 directly in terms of distance along the strut, with M_η entering only into the determination of the length $l(1-\zeta)$ of the yielding zone.

2·3. *Assumed initial curvature*

The initial curvature enters the analysis in the expression for the minor-axis bending moment M_η, namely,

$$M_\eta = B_2\left(\frac{d^2u}{dz^2}-\frac{d^2u_0}{dz^2}\right), \tag{27}$$

and since we are concerned with conditions near instability, we shall use the approximation (18) that the fundamental mode only is of importance, so that

$$M_\eta = (k_1-1)\,B_2\frac{d^2u_0}{dz^2}, \tag{28}$$

where k_1 is given by equation (21).

We now write

$$\left|B_2\frac{d^2u_0}{dz^2}\right| = K\frac{\psi(z_1)}{\psi_m}, \tag{29}$$

where ψ_m is the maximum value of ψ for varying z_1, so that K is the maximum value of the left-hand side of equation (29) and depends on B_2 and the magnitude of the initial imperfection. For the estimation of K, we first take $B_2 = B_y$ as a safe assumption since $B_2 \leqslant B_y$ everywhere in the strut; secondly, we assume that the greatest value of d^2u_0/dz^2 is the same as that in Horne's analysis of the problem, whence we obtain

$$K = 0{\cdot}0015\pi^2\frac{B_y r_y}{a_y l} = 0{\cdot}0015\pi^2 EZ_y\left(\frac{r_y}{l}\right). \tag{30}$$

As defined in equation (29), the function $\psi(z_1)$ is indeterminate to the extent of a multiplying constant, but this indeterminacy is of no importance, since ψ affects the final results only through the ratio ψ/ψ_m. Now, from (29) and (18), the form of $\psi(z_1)$ will depend on B_2/B_y, as given by equation (26) for $M = M_p$, and on the shape of the fundamental mode u_1. An exact solution for u_1 has not been obtainable and approximations regarding ψ will be necessary, but their detailed consideration is lengthy and will be deferred to appendix B. For the immediate analysis, it suffices to assume that ψ is a known function of z_1 for given values of β and ζ when $M = M_p$.

2·4. *The criterion for $M = M_p$ prior to instability* ($\beta \geqslant 0$)

If the forms (23) and (26) are used for the respective rigidities C and B_2, the factors H_1, H_2 in equations (20) and (21) can be determined as discussed in appendix A, where H_1 is a function of ζ only while H_2 is a function of both ζ and β. These factors may therefore be assumed known in the main analysis.

On the earlier assumptions, the section $z_1 = \zeta$ which is just yielding when $M = M_p$ will be governed by the equation

$$f_L = \frac{P}{A}+\frac{M_p}{Z_x}[\zeta+\beta(1-\zeta)]+\frac{|M_\eta|}{Z_y} \quad (z_1 = \zeta), \tag{31}$$

where from (28) and (29), $\quad |M_\eta| = (k_1-1)\,K[\psi(\zeta)/\psi_m] \quad (32)$

and K is given by (30). Also, from (21) with $M = M_p$ we can write

$$k_1 - 1 = \gamma_y/(1-\gamma_y), \tag{33}$$

where
$$\gamma_y = \frac{P}{P_0 H_1(\zeta)} + \frac{M_p^2}{M_0^2 H_2(\zeta,\beta)} \tag{34}$$

and we note that $k_1 > 1$ and $0 < \gamma_y < 1$.

We now introduce the notation

$$\left.\begin{aligned} M_L &= (f_L - p)\,Z_x, \\ p &= P/A, \end{aligned}\right\} \tag{35}$$

so that M_L is the moment at first yield of a section subjected to an axial stress p and a bending moment about the major axis only. Hence, from equation (31) with use of (32), (33) and (35), we obtain

$$M_L = M_p[\zeta + \beta(1-\zeta)] + K\frac{Z_x}{Z_y}\left(\frac{\gamma_y}{1-\gamma_y}\right)\frac{\psi(\zeta)}{\psi_m}. \tag{36}$$

The terms in equation (36) which involve the slenderness ratio l/r_y are, first, K as given by (30), and secondly γ_y, which from (22) and (34) may be expressed in the form

$$\gamma_y = \alpha(\zeta)\,(l/r_y)^2, \tag{37}$$

where
$$\alpha(\zeta) = \frac{p}{\pi^2 E H_1(\zeta)} + \frac{1}{\pi^2 F(\beta)}\left[\frac{M_p^2}{ETZ_x^2}\right]\frac{1}{H_2(\zeta,\beta)}. \tag{38}$$

Hence, if we substitute (37) and (30) in (36) and rearrange, we find

$$\left[\frac{1}{\alpha(\zeta)}\frac{r_y}{l} - \frac{l}{r_y}\right]\left[\frac{M_L}{M_p} - \beta - (1-\beta)\,\zeta\right] = 0{\cdot}0015\pi^2\frac{EZ_x\psi(\zeta)}{M_p\psi_m}. \tag{39}$$

For a given strut and given values of p and β, equation (39) may be regarded as giving the value of ζ when the strut is in equilibrium under the axial stress p with end moments M_p and βM_p. Now the value of ζ is necessarily restricted to sections within the strut so that $1 \geqslant \zeta \geqslant 0$ and this permissible range is further curtailed by the condition that $|M_\eta|$ is positive in equation (31), whence with use of (35) it follows that

$$\left.\begin{aligned} 0 \leqslant \zeta &\leqslant \zeta_0, \\ \beta + (1-\beta)\,\zeta_0 &= M_L/M_p. \end{aligned}\right\} \tag{40}$$

where

Here it may be noted that $\zeta = \zeta_0$ gives the limit of the yielding zone when $M = M_p$ for a straight strut, so that $\zeta < \zeta_0$ simply expresses the physical expectation that when there is an initial curvature, the additional moment M_η causes the yielding to extend further down the strut than it does for the straight strut.

In order to interpret equation (39) further, it is necessary to consider qualitatively the behaviour of struts under standard loading in those cases where the struts fail by instability when $M_L < M < M_p$ after some yielding has occurred, even though the section of maximum yielding remains at the top end. We assume for these struts that the length of the yielding zone increases steadily with M for $M > M_L$ and that the equilibrium curve for M against the length of yielding zone is of the form shown by the curve ACB of figure 3. The essential features are that M rises steadily to a single maximum value $M_c < M_p$, and then decreases steadily with no subsequent rise to a further maximum value. This behaviour agrees with relevant experimental data and it implies that when instability occurs prior to $M = M_p$, the strut cannot be in equilibrium with $M = M_p$ for any permissible value of ζ. Further, if we consider a series of such struts differing only in slenderness-ratio l/r_y under the same standard loading conditions (i.e. the same p, β), it is reasonable to assume that the maximum moment M_c at instability will increase steadily as the slenderness ratio decreases until a critical slenderness ratio is reached for which $M_c = M_p$. Then, for further decrease of slenderness-ratio the fully-plastic moment will be always developed prior to instability and there will be at least one permissible value of ζ for which the strut is in equilibrium with $M = M_p$. On this basis, the condition that $M = M_p$ can be obtained without instability corresponds to the condition that equation (39) has one root at least for ζ in the permissible range $0 \leqslant \zeta \leqslant \zeta_0$.

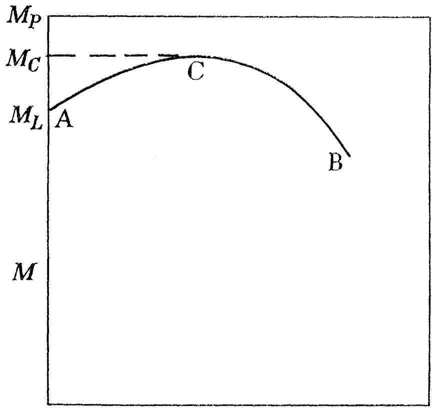

FIGURE 3. Qualitative variation of M with increasing yielding.

Now if we consider a given standard loading (p, β) on a series of struts which differ only in the slenderness-ratio l/r_y, then equation (39) is a quadratic equation for l/r_y and we should expect a maximum value of l/r_y in the range $0 \leqslant \zeta \leqslant \zeta_0$ corresponding to the critical case $M_c = M_p$ in the preceding discussion. Calculations for I-section struts, discussed later, confirm this expectation in giving a typical curve of the form DEF in figure 4 for the variation of l/r_y with ζ as derived from (39), noting in advance that the calculations necessarily exclude cases where yielding extends over the whole strut before both $M = M_p$ and instability. Figure 4 indicates that for stocky struts of small $l/r_y <$ OF there is a single permissible value of ζ when $M = M_p$, and the value of $\zeta_0 - \zeta$ is then small corresponding to only a small effect of M_η in extending the yielding zone for these stocky struts. For larger values of l/r_y, greater than OF but less than the maximum ordinate at E in figure 4, there are two values of ζ for a given slenderness-ratio. However, it is clear physically that the decrease in ζ as l/r_y increases will be a continuous process, and hence for continuity from the case of the stocky struts, it is clear that it is the larger value of ζ which is relevant when there are two roots of (39) for ζ in the permissible range. This larger value corresponds to a point on the portion DE of the curve DEF in figure 4, and the remaining portion EF is shown as a broken curve to denote that it is not physically relevant.

Summing up, equation (39) can be used to derive the portion DE of the curve DEF in figure 4 for given values of quantities other than l/r_y and ζ; the fully-plastic moment $M = M_p$

will then be attained at the top end without instability if the slenderness-ratio is less than the critical value corresponding to the maximum point E of this curve. This maximum value corresponds to a strut for which $M = M_p$ just at the onset of instability, and is one estimate of the critical slenderness-ratio which is the object of the analysis. This estimate is, however, based on the assumption that maximum yielding occurs at the top end for which the governing equation is the inequality (25). If this is violated, the assumed form (26) for B_2 is no longer reasonable since the incipient plastic hinge tends to occur away from the top end and, as already noted, instability will then always occur before $M = M_p$. We must therefore consider whether (25) further restricts the slenderness-ratio.

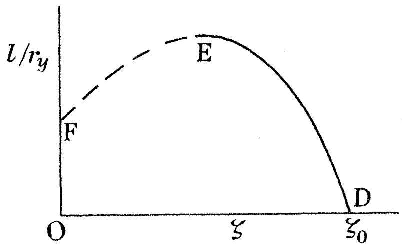

FIGURE 4. Variation of l/r_y with ζ from equation (39).

From equation (24), putting $M_\xi = M_p f(z)$ and using (6), (32) and (33), the inequality (25) becomes

$$(1-\beta)\frac{M_p}{Z_x}+\frac{K}{Z_y}\left(\frac{\gamma_y}{1-\gamma_y}\right)\frac{\psi'(1)}{\psi_m} \geqslant 0, \tag{41}$$

which may be simplified by using (36) and (40) to give the relation

$$1+(\zeta_0-\zeta)\,\psi'(1)/\psi(\zeta) \geqslant 0 \quad (\beta \neq 1). \tag{42}$$

If we now assume, as in appendix B, the specific form

$$\psi(z_1) = [z_1+\beta(1-z_1)]\,z_1(1-z_1^m), \tag{43}$$

the inequality (42) becomes simply

$$\psi(\zeta)/m+\zeta \geqslant \zeta_0, \tag{44}$$

in which m will be a function of ζ and β as discussed in appendix B. The later calculations show that $\zeta+\psi/m$ increases with ζ for a given β and hence we can conveniently write the inequality in the form

$$\left.\begin{aligned} &\zeta \geqslant \zeta_1, \\ \text{where}\qquad &\psi(\zeta_1)/m+\zeta_1 = \zeta_0, \end{aligned}\right\} \tag{45}$$

and the value of m refers to $\zeta = \zeta_1$.

If the condition (45) is now considered in relation to the earlier discussion of figure 4, we can distinguish two categories involving different forms for the criterion of the attainment of $M = M_p$ without instability. We use the notation $(l/r_y)_m$ and ζ_m to denote values at the maximum point E of the curve DEF in figure 4, and the notation $(l/r_y)_{\zeta_1}$ to denote the value of (l/r_y) given by equation (39) when $\zeta = \zeta_1$. The first category comprises those strut and loading conditions such that $\zeta_1 \leqslant \zeta_m$ so that the inequality (45) is satisfied by the whole portion DE of the curve DEF in figure 4. In this category the relevant criterion is

$(l/r_y) \leqslant (l/r_y)_m$. The second category comprises conditions where $\zeta_1 > \zeta_m$ so that only the portion $\zeta_1 \leqslant \zeta \leqslant \zeta_0$ of DE in figure 4 satisfies the inequality (45). In this category the relevant criterion is $(l/r_y) \leqslant (l/r_y)_{\zeta_1}$. The two categories correspond to two conceivable physical types of instability. The first type refers to conditions in which there is no tendency for a plastic hinge to form away from the top end, and the potential instability is then essentially similar to that of an elastic strut of the non-uniform rigidity given by (26). The second type refers to conditions where a plastic hinge is tending to form away from the ends before $M = M_p$ and thus form a lateral mechanism with inevitable prior instability.

The final criterion for the development of a fully plastic moment at the top end without instability may be expressed in the form

$$\left.\begin{aligned} \frac{l}{r_y} &\leqslant \left(\frac{l}{r_y}\right)_m \quad \text{if} \quad \zeta_1 \leqslant \zeta_m, \\ \frac{l}{r_y} &\leqslant \left(\frac{l}{r_y}\right)_{\zeta_1} \quad \text{if} \quad \zeta_1 \geqslant \zeta_m, \end{aligned}\right\} \tag{46}$$

where equality signs correspond to the critical slenderness ratio such that instability is just about to occur when $M = M_p$.

For the evaluation of the critical slenderness-ratio, equation (39) may be written as a quadratic equation for l/r_y in the form

$$\left(\frac{l}{r_y}\right)^2 + \mu(\zeta)\frac{l}{r_y} - \frac{1}{\alpha(\zeta)} = 0, \tag{47}$$

where $\alpha(\zeta)$ is defined by (38), while with use of (40),

$$\mu(\zeta) = 0{\cdot}0015\pi^2 \frac{EZ_x}{M_p}\left(\frac{\psi(\zeta)}{\psi_m}\right)\frac{1}{(1-\beta)(\zeta_0-\zeta)}. \tag{48}$$

The roots of the quadratic equation (47) for l/r_y will always be real and of opposite sign since $\alpha > 0$, but only the positive root is of physical significance, namely,

$$\frac{l}{r_y} = -\frac{\mu}{2} + \sqrt{\left(\frac{1}{\alpha} + \frac{\mu^2}{4}\right)}. \tag{49}$$

The basic equations for determining the critical slenderness ratio are first, (49) in which α and μ are given by (38) and (48), respectively, and secondly, (45) which defines ζ_1. Subsidiary equations for evaluating H_1, H_2, and ψ are given in appendices A and B.

Before considering the numerical evaluations for I-sections, one general restriction on the range of validity of the calculations should be noted. Thus, the analysis involves the assumption that part of the strut remains elastic when $M = M_p$; this condition corresponds to $\zeta \geqslant 0$ and from (40) it is clearly a necessary condition that $\zeta_0 \geqslant 0$, and hence that

$$\beta \leqslant M_L/M_p. \tag{50}$$

Loading conditions where $\beta > M_L/M_p$ are not covered by the present analysis and we note only that they are either cases of relatively high axial loading for which M_L/M_p is small, or cases of nearly uniform moment where $\beta = 1$; in either case $M = M_p$ can only be attained without instability for relatively very short struts.

2·5. *Evaluation of the critical slenderness ratio for steel* I-*sections* ($\beta \geqslant 0$)

The values of both M_L and M_p depend on the axial stress and following Horne (1956) we write

$$n = p/f_L \tag{51}$$

and assume that for practical I-sections,

$$\left.\begin{aligned} M_p/f_L Z_x &= 1{\cdot}15(1-1{\cdot}5n^2) \qquad (0 \leqslant n \leqslant 0{\cdot}4),\\ M_p/f_L Z_x &= \tfrac{1}{9}\ 1{\cdot}15(1-n)\,(11+n) \quad (0{\cdot}4 \leqslant n \leqslant 1),\end{aligned}\right\} \tag{52}$$

while we note from (35) and (51) that

$$M_L/f_L Z_x = 1-n. \tag{53}$$

Thus the shape factor M_L/M_p depends only on n and we write

$$M_L/M_p = S(n), \tag{54}$$

where

$$\left.\begin{aligned} S(n) &= \frac{1-n}{1{\cdot}15(1-1{\cdot}5n^2)} \qquad (0 \leqslant n \leqslant 0{\cdot}4),\\ S(n) &= \frac{9}{1{\cdot}15(11+n)} \qquad (0{\cdot}4 \leqslant n \leqslant 1).\end{aligned}\right\} \tag{55}$$

Numerical values for $S(n)$ are given in table 1 for $n = 0(0{\cdot}1)1$.

TABLE 1. VALUES OF $S(n)$

n	$S(n)$	n	$S(n)$	n	$S(n)$
0	0·870	0·4	0·687	0·8	0·663
0·1	0·795	0·5	0·680	0·9	0·658
0·2	0·740	0·6	0·675	1·0	0·652
0·3	0·704	0·7	0·669		

Equations (48) and (38) may now be written in the forms

$$\mu(\zeta) = 0{\cdot}0015\pi^2 \frac{E}{f_L}\left(\frac{S(n)}{1-n}\right) \frac{\psi(\zeta)}{(1-\beta)\,(\zeta_0-\zeta)\,\psi_m}, \tag{56}$$

$$\alpha(\zeta) = \frac{nf_L}{\pi^2 EH_1(\zeta)} + \frac{1}{\pi^2 F(\beta)\,H_2(\zeta,\beta)}\left(\frac{f_L^2}{ET}\right)\left(\frac{1-n}{S(n)}\right)^2, \tag{57}$$

where from (40) and (54)

$$\zeta_0 = [S(n)-\beta]/(1-\beta), \tag{58}$$

and we note that the permissible range for ζ is $0 \leqslant \zeta \leqslant \zeta_0$. Values of ζ_0 are given in table 2.

TABLE 2. VALUES OF ζ_0

$\beta \backslash n$	0	0·1	0·2	0·4	0·6	0·8
0	0·870	0·795	0·740	0·687	0·675	0·663
0·2	0·837	0·744	0·675	0·608	0·593	0·579
0·4	0·783	0·658	0·567	0·478	0·458	0·439
0·6	0·674	0·489	0·350	0·216	0·187	0·158
0·8	0·348					

The choice of shape for $\psi(z_1)$ is considered in appendix B and we note that the choice is such that $\psi(\zeta)/\psi_m$ can be tabulated as a function of ζ and β. Thus equations (56) and (57) for use in (49) involve ζ and the following non-dimensional parameters depending on the loading and section properties

$$n, \quad \beta, \quad f_L/E, \quad f_L/T. \tag{59}$$

The numerical evaluation of $(l/r_y)_m$ for use in the criterion (46) involves the following steps:

(*a*) Evaluation of H_1 as a function of ζ; it is independent of the parameters (59). This evaluation is considered in appendix A with the results given in table 3 and figure 5.

TABLE 3. VALUES OF $H_1(\zeta)$ FOR STRUT WITH ENDS SIMPLY SUPPORTED ABOUT THE MINOR AXIS

ζ	$H_1(\zeta)$	ζ	$H_1(\zeta)$
1	1	0·379	0·592
0·840	0·986	0·278	0·514
0·755	0·951	0·165	0·446
0·660	0·877	0·098	0·412
0·566	0·779	0	0·372
0·450	0·657		

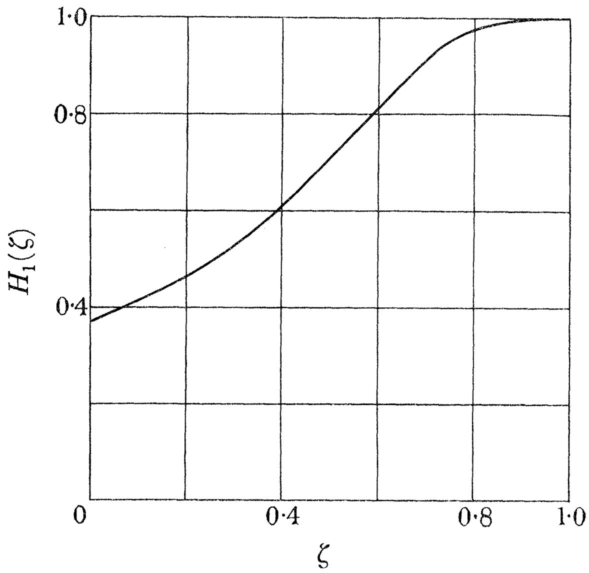

FIGURE 5. $H_1(\zeta)$ for strut with ends simply supported about the minor axis.

(*b*) Evaluation of FH_2 as a function of ζ and β; it is independent of the parameters other than β in (59). This evaluation is considered in appendix A with results given in table 4.

(*c*) Evaluation of $\alpha(\zeta)$ as a function of ζ; it involves all the parameters (59).

(*d*) Choice of shape for ψ (appendix *B*) and the evaluation of $\mu(\zeta)$ which involves all the parameters (59) except f_L/T.

(*e*) Evaluation of l/r_y as a function of ζ from (49) for given values of the parameters (59), and the determination of its maximum value for $0 \leqslant \zeta \leqslant \zeta_0$.

The associated determination of ζ_1 is considered in appendix B, with the results given in table 5. It may be noted that ζ_1 is a function only of n and β on the assumptions made in appendix B. The evaluation of $(l/r_y)_{\zeta_1}$ then follows from (49). Since ζ_1 has to be determined in all cases to check which part of the criterion (46) is operative, and since $(l/r_y)_{\zeta_1}$ involves only the one value of ζ whereas $(l/r_y)_m$ involves varying ζ, it is simpler to find $(l/r_y)_{\zeta_1}$ first and then check whether $\zeta_1 < \zeta_m$ or $\zeta_1 > \zeta_m$. This check can then most easily be made by evaluating l/r_y for a value $\zeta < \zeta_1$ close to ζ_1. Then, referring to figure 4, it is clear that if this new value of l/r_y is greater than $(l/r_y)_{\zeta 1}$, then ζ_1 lies on the portion DE of the curve DEF and the operative criterion is $(l/r_y)_{\zeta 1}$. Conversely, if the new value of l/r_y is less than $(l/r_y)_{\zeta 1}$, then ζ_1 lies on the portion EF and the operative criterion is $(l/r_y)_m$. It is only in the second case that the more extensive calculations of $(l/r_y)_m$ are strictly necessary. However, though not finally needed for the criterion, sufficient calculations of (l/r_y) for varying ζ have been carried out to verify the shape shown for the curve DEF in figure 4.

TABLE 4. VALUES OF $100/\pi^2 FH_2$ FOR A STRUT WITH ENDS SIMPLY SUPPORTED ABOUT THE MINOR AXIS

β \ ζ	0	0·2	0·4	0·6	0·7	0·8	1·0
0	14·2	11·4	8·55	5·80	4·59	3·70	3·20
0·2	16·1	12·9	9·70	6·65	5·40	4·55	4·10
0·4	18·3	14·6	11·0	7·70	6·45	5·60	5·20
0·6	20·8	16·7	12·6	9·00	7·70	6·90	6·50
0·8	23·8	19·1	14·5	10·6	9·25	8·45	8·15
1·0	27·2	21·8	16·6	12·4	11·0	10·3	10·0

TABLE 5. VALUES OF ζ_1 FOR A STRUT WITH ENDS SIMPLY SUPPORTED ABOUT THE MINOR AXIS

β \ n	0	0·1	0·2	0·4	0·6	0·8
0	0·773	0·708	0·663	0·620	0·610	0·600
0·2	0·725	0·647	0·592	0·537	0·524	0·512
0·4	0·656	0·554	0·483	0·414	0·397	0·382
0·6	0·547	0·402	0·290	0·181	0·157	0·134
0·8	0·272	—	—	—	—	—

2·6. *Scope of numerical calculations*

Loading parameters β and n. Attention has been concentrated on the case $\beta \geqslant 0$ and the following combinations of β and n have been covered

$$\left.\begin{aligned}\beta &= 0(0{\cdot}2)0{\cdot}6,\\ n &= 0,\ 0{\cdot}1,\ 0{\cdot}2(0{\cdot}2)0{\cdot}8,\end{aligned}\right\}$$

$$\left.\begin{aligned}\beta &= 0{\cdot}8,\\ n &= 0,\end{aligned}\right\}$$

where the calculations for $\beta = 0{\cdot}8$ have been restricted to $n = 0$ in view of the condition (50), noting that $M_L/M_p = S(n)$ as given in table 1.

Material and section parameters f_L/E and f_L/T. Values appropriate to steel I-sections have been chosen and for easy comparison with Horne's results (1956) we have taken

$$E = 13\,000 \text{ tons/sq.in.},$$
$$f_L = 15{\cdot}25 \text{ tons/sq.in.},$$
$$T = 24,\ 40,\ 100,\ 400 \text{ tons/sq.in.}$$

2·7. *Comparison with Horne's results*

The final results obtained for the critical slenderness ratio, such that $M = M_p$ is attained without instability for all smaller slenderness ratios, are listed as unbracketed values in table 6. It should be noted that in all but one case, namely, $T = 400$, $n = 0$, $\beta = 0$, the value given refers to $(l/r_y)_{\zeta_1}$, with $\zeta_1 > \zeta_m$ in all but this one case. Even in this case, the quoted value $(l/r_y)_m = 608$ is only slightly greater than the corresponding $(l/r_y)_{\zeta_1} = 605$ which occurs for $\zeta_1 < \zeta_m$. Hence, over virtually the whole range of the calculations, the operative criterion is that a plastic hinge does not tend to form away from the top end prior to $M = M_p$, rather than the other criterion of avoiding instability due to the general decrease in rigidity over the yielded portion with maximum yielding at the top end. In general, the values are shown to three significant figures to avoid cumulative errors in any interpolation of the results, but it will be realized that the assumptions in the analysis do not warrant individual results being regarded as accurate to three figures for practical application.

TABLE 6. CRITICAL SLENDERNESS RATIO FOR A STRUT WITH ENDS SIMPLY SUPPORTED ABOUT THE MINOR AXIS

Note. The unbracketed value of the slenderness ratio is that given by the present theory. The first bracketed value is that given by Horne's earlier theory (1956). The second bracketed value is that given by Horne's later theory (1964); the values in bold figures are cases where Horne's limiting curve is the operative criterion in the later theory.

		T = 24 tons/sq.in. l/r_y			T = 40 tons/sq.in. l/r_y			T = 100 tons/sq.in l/r_y			T = 400 tons/sq.in. l/r_y
β	n										
0	0	115	(130)	(140)	160	(171)	(172)	279	(290)	(270)	608
	0·1	86·5	(105)	(106)	112	(130)	(113)	155	(165)	(140)	200
	0·2	70·1	(87)	(82)	86·3	(98)	(77)	110	(111)	(88)	129
	0·4	51·5	(58)	(48)	58·2	(63)	(41)	66·2	(62)	(44)	70·8
	0·6	34·8	(37)	(31)	36·6	(38)	**(32)**	38·6	(36)	**(33)**	39·6
	0·8	16·3	(16)	**(27)**	16·5	(17)	**(28)**	16·8	(16)	**(29)**	17·0
0·2	0	91·3	(109)	(120)	129	(142)	(147)	231	(246)	(233)	512
	0·1	68·5	(86)	(87)	91·4	(110)	(88)	134	(140)	(110)	182
	0·2	55·4	(64)	(58)	70·4	(76)	(52)	94·4	(87)	(60)	115
	0·4	41·2	(30)	**(36)**	47·8	(33)	**(38)**	55·9	(34)	**(41)**	61·2
	0·6	28·1	(14)	**(31)**	30·0	(16)	**(32)**	31·9	(15)	**(33)**	33·0
	0·8	13·2	(5)	**(27)**	13·4	(5)	**(28)**	13·6	(5)	**(29)**	13·7
0·4	0	68·4	(86)	(96)	99·5	(120)	(118)	185	(204)	(195)	426
	0·1	49·1	(58)	(55)	68·0	(76)	**(61)**	107	(100)	**(72)**	155
	0·2	39·4	(20)	**(44)**	52·0	(25)	**(48)**	73·8	(30)	**(55)**	94·2
	0·4	29·2	(0)	**(36)**	34·8	(0)	**(38)**	42·1	(0)	**(41)**	47·1
	0·6	19·6	(0)	**(31)**	21·3	(0)	**(32)**	22·9	(0)	**(33)**	23·9
	0·8	9·2	(0)	**(27)**	9·4	(0)	**(28)**	9·5	(0)	**(29)**	9·7
0·6	0	42·0	(46)	**(66)**	64·2	(70)	**(84)**	129	(141)	**(131)**	325
	0·1	29·2	(0)	**(52)**	42·3	(0)	**(61)**	72·0	(0)	**(72)**	114
	0·2	22·8	(0)	**(44)**	31·3	(0)	**(48)**	47·4	(0)	**(55)**	64·2
	0·4	16·5	(0)	**(36)**	20·2	(0)	**(38)**	25·3	(0)	**(41)**	29·0
	0·6	11·2	(0)	**(31)**	12·2	(0)	**(32)**	13·4	(0)	**(33)**	14·1
	0·8	5·2	(0)	**(27)**	5·3	(0)	**(28)**	5·4	(0)	**(29)**	5·4
0·8	0	14·9	(0)	**(66)**	24·3	(0)	**(84)**	56·0	(0)	**(131)**	176

Table 6 lists in brackets the corresponding results from Horne's 1956 and 1964 theories. These have been read from the curves in Horne 1956, 1964*b* and are liable to errors of order ± 1 in such reading. In each case, the first bracketed value relates to Horne's 1956 theory, and the second bracketed value relates to Horne's 1964 theory. No values are shown for $T = 400$ which lies outside the range of Horne's calculations. The values for the 1964 theory refer to a yield stress of 16 tons/sq.in. as compared to 15·25 tons/sq.in. for Horne's earlier theory and the present theory. However, this 5 % change of yield stress will of itself lead only to a somewhat smaller change in the critical slenderness ratio, and is unimportant compared to much larger differences between the three sets of theoretical results in table 6.

It is seen from table 6 that Horne's earlier theory in comparison with the present theory, gives somewhat higher values of the critical slenderness ratio for the smaller values of β and n, and lower slenderness ratios for the larger values of β and n. The difference is most pronounced for $\beta \geqslant 0{\cdot}4$, $n \geqslant 0{\cdot}4$, where Horne's earlier theory definitely errs on the safe side by excluding any length of strut. On the other hand, it may be noted that for $\beta = 0$ and $T = 40$ to 100, the results from the two theories are in fair agreement, especially for $T = 100$. The disagreement between the present theory and Horne's later theory is in general larger than that between the present theory and Horne's earlier theory. In particular, the nature of the disagreement for the high values of β and n is completely reversed, in that Horne's later theory gives larger slenderness ratios than the present theory. In fact, for the majority of the combinations of β, n and T in table 6, the present theory gives values intermediate between those from Horne's two theories. The big differences between Horne's earlier and later results arise mainly from two additional features in his later theory. First, he introduces an additional and overriding criterion which implies that a fully plastic hinge may be assumed to occur for any combination of β, n and T for sufficiently short struts. This leads to the complete reversal of the earlier result that plastic-hinge action could not be assumed for any strut, however short for the higher values of β and n. Secondly, Horne's later theory includes allowances for warping rigidity and the Wagner effect, which were neglected in his earlier theory as they are in the present theory. Surprisingly, the inclusion of warping rigidity leads in most cases to a decrease in the critical slenderness ratio, which implies that neglect of warping rigidity is an unsafe assumption. This conclusion seems contrary to physical intuition and is probably a spurious result arising from the oversafe manner in which the effect of warping rigidity has been included in Horne's later theory (see Fox 1965).

On a point of detail, the effect of yielding on the stability of a straight strut is allowed for by the factors H_1 and H_2 in the present analysis and by the factor H in Horne's analyses. Values of H_2 can be derived from table 4 and Horne's table of F and it is found that H_2 decreases as β decreases for a given ζ and that $H_2(\zeta, \beta) < H_1(\zeta)$ for $\beta < 1$. Thus, for the smaller values of ζ_1 in table 5, reference to figure 5 shows that both H_1 and H_2 are less than Horne's safe value $H = 0{\cdot}7$. It must therefore be emphasized that Horne's lower bound $H = 0{\cdot}7$ is not a general result applying to the actual loading case, but is justified only in association with his assumption that instability corresponds to yield spreading down to the centre of the strut in his equivalent uniform-moment problem.

3. Extension of the theory to the case of a strut clamped about the minor axis at its ends

The case where the strut is clamped at its ends about the minor axis has also been analysed on the same general assumptions and method of analysis as for the case of simple support about the minor axis. We are again concerned with conditions such that M_p is attained at the top end and since we assume that $B_2 = 0$ at this end when $M = M_p$, it follows that the effective end condition is one of simple support about the minor axis at this end, even though it may have been clamped initially prior to loading. The essential change is therefore in the boundary condition for u at the bottom of the strut which leads to consequential changes in some of the earlier equations which were derived for simple support about the minor axis. The equations which are changed are given below, where in each case we have added a to the number of the original equation so that, for example, equation (4a) is the modification of equation (4) when the bottom is clamped instead of simply supported about the minor axis. The revised end conditions are

$$\left.\begin{aligned} u = u_0 &= 0 \qquad (z=0,\ z=l),\\ \frac{\mathrm{d}u}{\mathrm{d}z} = \frac{\mathrm{d}u_0}{\mathrm{d}z} &= 0 \qquad (z=0),\\ B_2\frac{\mathrm{d}^2u}{\mathrm{d}z^2} = B_2\frac{\mathrm{d}^2u_0}{\mathrm{d}z^2} &= 0 \qquad (z=l),\end{aligned}\right\} \tag{1a}$$

and the revised equation (4) is

$$B_2\left(\frac{\mathrm{d}^2u}{\mathrm{d}z^2}-\frac{\mathrm{d}^2u_0}{\mathrm{d}z^2}\right)+Pu = -\phi Mf(z)+(l-z)X, \tag{4a}$$

where X is the shear force in the direction of the major axis at the top of the strut, with Xl equal to the minor axis clamping moment at the bottom. Equations (4a) and (5) give

$$\frac{\mathrm{d}}{\mathrm{d}z}C\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{B_2}{f(z)}\left(\frac{\mathrm{d}^2u}{\mathrm{d}z^2}-\frac{\mathrm{d}^2u_0}{\mathrm{d}z^2}\right)+\frac{Pu}{f(z)}-\frac{(l-z)X}{f(z)}\right]+M^2f(z)\frac{\mathrm{d}^2u}{\mathrm{d}z^2}=0, \tag{7a}$$

which is subject to the end conditions (1a) plus the additional condition

$$lX = B_2\left(\frac{\mathrm{d}^2u}{\mathrm{d}z^2}-\frac{\mathrm{d}^2u_0}{\mathrm{d}z^2}\right) \qquad (z=0), \tag{60}$$

which follows from (4a) and (1a).

For the straight strut we put $u_0 = 0$, $u = u_n(z)$, $X = X_n$ to denote the typical eigenfunction solution governed by the modified equations:

$$\frac{\mathrm{d}}{\mathrm{d}z}C\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{B_2}{f(z)}\frac{\mathrm{d}^2u_n}{\mathrm{d}z^2}+\frac{Pu_n}{f(z)}-\frac{(l-z)X_n}{f(z)}\right]+M^2f(z)\frac{\mathrm{d}^2u_n}{\mathrm{d}z^2}=0, \tag{8a}$$

$$\left.\begin{aligned} u_n = \mathrm{d}u_n/\mathrm{d}z &= 0 \qquad (z=0),\\ u_n = B_2(\mathrm{d}^2u_n/\mathrm{d}z^2) &= 0 \qquad (z=l),\\ B_2(\mathrm{d}^2u_n/\mathrm{d}z^2) &= lX_n \qquad (z=0)\end{aligned}\right\} \tag{9a}$$

$$B_2(\mathrm{d}^2u_n/\mathrm{d}z^2)+Pu_n-(l-z)X_n = -\phi_nMf(z). \tag{10a}$$

The solution of these equations will correspond to a specific eigenrelation of the form (11), though the forms of F_n and u_n will, of course, differ in detail from those of the earlier simply supported case. As before, we consider a set of eigenfunctions corresponding to points on a radius vector in the (P, M^2) plane and the orthogonal relation (12) will again hold. The solution of (7a) is then sought by expansions of the forms (13) and (14) together with the expansion

$$X = \sum_{n=1}^{\infty}(k_n-1)\,a_nX_n \tag{61}$$

which follows from (13), (14), (60) and (9*a*). If we then substitute (13), (14) and (61) in (7*a*) and satisfy this equation for the typical *n*th terms, we find

$$\frac{\mathrm{d}}{\mathrm{d}z}C\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{B_2}{f(z)}\frac{\mathrm{d}^2u_n}{\mathrm{d}z^2}+\frac{k_nPu_n}{(k_n-1)f(z)}-\frac{(l-z)X_n}{f(z)}\right]+\frac{M^2k_nf(z)}{k_n-1}\frac{\mathrm{d}^2u_n}{\mathrm{d}z^2}=0, \qquad (15a)$$

with associated end conditions (9*a*). But equation (15*a*) is the same as (8*a*) except that $k_nP/(k_n-1)$ replaces P while $k_nM^2/(k_n-1)$ replaces M^2. Hence we again have a solution of the form (16) with a geometrical interpretation in terms of figure 1 and equation (17). We then use the approximation (18) that the fundamental mode is dominant and adopt the safe approximation (19) for the eigenrelation $F_1=0$; but P_{10} and M_{10} now refer to the strut clamped about the minor axis at the foot. However, we can still write (19) in the form (20) leading to (21), with P_0, M_0 still given by (22), provided H_1, H_2 now allow both for the effect of yielding and the effect of clamping the foot about the minor axis. On this definition of H_1, H_2, the further analysis for the clamped foot proceeds in the same way as for the simply supported foot, with equations (22) to (42) applying equally to both cases. The form of $\psi(z_1)$ must however be changed to allow for the foot clamping, so that equations (43) to (45) require modification; this is discussed in appendix B. The remainder of the earlier main analysis from equation (46) onwards is applicable in unchanged form to the new foot condition and the general method of calculating the critical slenderness ratio is also unchanged. The essential difference is that the relevant numerical values of H_1, H_2, $\psi(\zeta)/\psi_m$ and ζ_1 are changed, leading of course to changed values of the critical slenderness ratio for given n, β and T. The method of calculating the modified values of H_1, etc. is discussed in appendices A and B.

The final results for the critical slenderness ratio are given in table 7 for the case of the ends clamped about the minor axis. All the results refer to $(l/r_y)_{\zeta_1}$ since the operative criterion is found to be that a plastic hinge does not tend to form away from the top end prior to $M=M_p$. No simple general relation has been found between these results and those of table 6 for the case of simple support about the minor axis. Thus, if we use R to denote the ratio of the critical slenderness ratio of table 6 to that of table 7 for the same β, n and T, then the effect of the clamping for a strut of length l could be expressed as a reduction of length to an equivalent length Rl with simple support about the minor axis. But the value of R in fact varies appreciably with β and n from about 0·7 for $\beta=0$, $n=0{\cdot}8$ to nearly unity for $\beta=0{\cdot}6$, $n=0{\cdot}8$ and the dependence of R on β, n, T does not appear to be simple. Hence there is no value in using the concept of an equivalent length to allow for minor-axis clamping in the present problem.

4. Comparison of theory and experiment

Various reports have been published on tests of steel I-section members under combined axial load and bending moment. Unfortunately, most of these tests cannot be usefully compared with the present theory, since in some cases the reports do not give all the relevant information, while in other cases the tests refer to conditions $\beta=\pm 1$ which lie outside the scope of the present theoretical results. Hence, reliable comparisons have been found possible for only relatively few tests, all of which relate to the case $\beta=0$.

Experiments by Horne, Gilroy, Neile & Wilson (1956) on model-scale struts included three tests for $\beta = 0$ with the ends simply supported about the minor axis, so that the relevant theoretical results are those of table 6. The value of T was approximately 40 tons/sq.in. and the full plastic moment was attained without instability in all three tests. Table 8 shows the experimental values of the slenderness ratio and the value of n at maximum load,

Table 7. Critical slenderness ratio for a strut with ends clamped about the minor axis

β	n	$T = 24$ tons/sq.in. l/r_y	$T = 40$ tons/sq.in. l/r_y	$T = 100$ tons/sq.in. l/r_y	$T = 400$ tons/sq.in. l/r_y
0	0	128	180	315	690
	0·1	99·1	131	194	270
	0·2	82·7	105	142	177
	0·4	64·5	75·4	89·3	98·7
	0·6	46·9	50·2	53·8	55·9
	0·8	23·0	23·3	23·8	24·0
0·2	0	105	150	271	611
	0·1	79·5	108	168	244
	0·2	64·7	84·7	119	153
	0·4	49·8	59·3	71·9	80·6
	0·6	35·2	38·2	41·4	43·2
	0·8	17·1	17·4	17·7	17·9
0·4	0	83·3	122	228	529
	0·1	60·4	84·6	137	207
	0·2	48·8	65·4	95·5	126
	0·4	36·5	44·3	54·8	62·4
	0·6	26·2	28·6	31·2	32·7
	0·8	12·5	12·7	13·0	13·1
0·6	0	51·5	79·5	161	413
	0·1	35·2	51·6	90·2	148
	0·2	25·3	35·3	55·3	77·6
	0·4	17·1	21·2	27·3	31·8
	0·6	11·7	13·0	14·3	15·1
	0·8	5·5	5·6	5·7	5·8
0·8	0	16·5	26·5	62·9	206

Table 8. Comparison of theory and experiment for members simply supported about the minor axis ($\beta = 0$, $T = 40$ tons/sq.in.)

		experimental		theoretical critical l/r_y		
data source	specimen no.	n	l/r_y	present theory	Horne (1956)	Horne (1964)
Horne *et al.* (1956)	HS 17	0·27	50·1	75	83	59
	HS 17/20	0·10	100·8	112	130	113
	HS 18	0	152·1	160	171	172
Heyman (1957)	HS 2	0·17	198	92	108	85
	HS 7	0·29	99	72	81	55

together with theoretical estimates of the critical slenderness ratio given by the present theory and by Horne's theories for $\beta = 0$, $T = 40$ tons/sq.in. These theoretical estimates are based on the value $f_L = 15{\cdot}25$ tons/sq.in. for the first two theories and the value $f_L = 16$ tons/sq.in. for Horne's (1964*b*) theory. These differ from the mean experimental value $f_L = 15{\cdot}44$ ton/sq.in. from bending tests but the differences are too small to affect the conclusion from table 8 that the theoretical estimates of critical slenderness ratio are all greater than the corresponding experimental values. Hence, all three theories are in agreement with the experimental result that the full plastic moment was attained without instability in each of the three tests. Heyman (1957) has reported similar tests using the same section of model-scale strut. These tests include two cases for $\beta = 0$ which are listed in table 8, together with the theoretical estimates corresponding to the experimental values of n at maximum load. Differences between experimental values of f_L and those used in the theories are too small to affect the essential conclusion from table 8 that the theoretical estimates are all appreciably less than the corresponding actual slenderness ratios. Hence, for these two tests, all three theories predict instability prior to $M = M_p$ and this occurred in both tests.

Summing up, for members simply supported about the minor axis, all three theories are in agreement with the five tests for which a reliable comparison has been found possible. Much more data are needed to distinguish between the merits of the different theories.

VanKuren & Galambos (1961) have summarized results of tests on wide-flanged steel beam columns, including five tests with $\beta = 0$ under conditions similar to those assumed in the present theory for members clamped about the minor axis at each end. The relevant theoretical results are those of table 7 and the theory is compared with experiment in table 9. It is seen that the theory agrees with experiment in four tests but errs on the unsafe side for the remaining test T 23. The values of n in table 9 are based on the experimental yield stress which varied from 16·7 to 17·8 tons/sq.in. in the five tests, whereas the theoretical estimates of slenderness ratio are based on $f_L = 15{\cdot}25$ tons/sq.in. If the experimental values of the yield stress are used in the theory, the theoretical estimates are all reduced by amounts of 13% or less and (*a*) the agreement for T 1, T 2 and T 31 is not significantly affected, (*b*) the revised theoretical estimate for T 13 is 94 so that the agreement becomes borderline with a small error on the safe side, and (*c*) the disagreement for T 23 is reduced with a revised theoretical estimate of about 163 for the critical slenderness ratio. The remaining disagreement may be due to the initial imperfection for T 23 being larger than that assumed in the theory. Thus, if the assumed imperfection were doubled the estimated slenderness ratio for T 23 would be reduced by about 25% and the disagreement with experiment would be removed.

Table 9. Comparison of theory with tests by VanKuren & Galambos (1961)

test no.	section	T (ton/sq.in.)	n	l/r_y expt	critical l/r_y theory	experimental result	theoretical prediction
T 1	8 WF 31	30	0·130	36	105	M_p attained	M_p attained
T 13	8 WF 31	30	0·122	96	106	M_p attained	M_p attained
T 2	8 WF 40	51	0·148	36	130	M_p attained	M_p attained
T 23	4 WF 13	93	0·114	140	180	Unstable	M_p attained
T 31	4 WF 13	93	0·122	194	173	Unstable	Unstable

Augusti (1964) has carried out model-scale tests of I-section members for $\beta = 0$ under the conditions assumed in the theory for a member clamped about the minor axis at its ends. He conducted two series of tests; in series I, the section had nominal dimensions of overall depth $\frac{3}{4}$ in., flange width $\frac{1}{4}$ in., flange thickness $\frac{1}{8}$ in. and web thickness $\frac{1}{16}$ in., while in series H the flange width was $\frac{1}{2}$ in. with other dimensions unchanged from series I. He tested several different lengths in each series, and for each length he carried out tests with different axial loads so that, for all but one length (H 8), he was able to bracket the critical slenderness ratio at which the failure changed from excessive major-axis bending with $M = M_p$ to collapse by lateral instability. These tests are thus particularly suitable for checking the present theory and the comparison is most easily made by comparing the theoretical and experimental values of n corresponding to a given critical slenderness ratio. The experimental values are not precise; instead, for a given length and section of specimen there is a range of n between the largest n for which failure was by excessive bending and the smallest n for which collapse occurred by lateral instability. The value of n for which the slenderness ratio is critical then lies somewhere in this range.

TABLE 10. COMPARISON OF THEORY WITH TESTS BY AUGUSTI (1964)

test series	span (in.)	l/r_y	critical n experimental	critical n theoretical
H	8	61·5	> 0·64	0·57
	16	123	0·48–0·57	0·33
	22	169	0·23–0·31	0·22
I	8	133	0·21–0·33	0·29
	12	200	0·19–0·29	0·17
	16	267	0·075–0·16	0·10
	22	367	0–0·051	0·052

The relevant theoretical results are those of table 7 and the comparison is given in table 10 which shows first the experimental range for the critical value of n and secondly the theoretical estimate of this value for the experimental slenderness ratio. Only three tests were carried out for the H specimens of span 8 in. and as they all failed by excessive bending only a lower bound to the critical value of n is given by these experiments. The calculated values of T were 360 tons/sq.in. for series I and 380 tons/sq.in. for series H, but since estimates of torsional rigidity cannot be very precise, it was considered sufficient to use the theoretical results for $T = 400$ tons/sq.in. for the comparison in table 10.

Table 10 shows that the theory errs on the safe side for all three lengths of series H, since the theoretical estimate in each case lies below the experimental lower bound for the critical value of n. Secondly, for I 8, I 12 and I 16, the theoretical estimates of critical n lie within or below the experimental bracket, indicating that the theory either agrees with experiment within the experimental bracket, or errs on the safe side. But for I 22, the theoretical estimate lies just above the experimental bracket for the critical n and thus errs on the unsafe side. However, the experimental values of n are based on experimental values for the yield stress ranging between 17·6 and 21·4 tons/sq.in. as compared to the value 15·25 tons/sq.in. used in the theory. If the experimental values of f_L are used in the theory, the theoretical estimates of critical n in table 10 are reduced variously by 0·01 to 0·03; in particular, the estimate for I 22 is reduced to $n = 0·035$ which lies within the experimental bracket. Hence, the use of the theory for Augusti's tests leads either to safe estimates or to estimates lying within the corresponding experimental brackets. The theory appears to be safer for series H than for series I; this may be due to the fact that warping rigidity, neglected in the theory, is relatively more important for the wider flange specimens of series H.

Summing up, for members clamped about the minor axis at each end, the agreement between theory and experiment is reasonably satisfactory with a tendency for the theory to err on the safe side. Exceptionally, for test T 23 of VanKuren & Galambos the theory errs significantly on the unsafe side; this error is reduced if allowance is made for the actual yield stress being higher than that assumed in the theory, and the disagreement could be completely removed if the assumed theoretical magnitude of the initial imperfection were doubled. The choice of this magnitude is possibly the most uncertain feature of the present theory and it may well need modification in the light of further experimental data which are certainly needed before any theory can be verified with confidence.

5. The variation of the critical slenderness ratio with varying yield stress

In the comparison of theory and experiment, the quoted effects of the difference between f_L and 15·25 tons/sq.in. were obtained from subsidiary calculations using the experimental values of f_L in the theory. Here we note a simpler correction for f_L which errs on the safe side in underestimating the critical slenderness ratio. Thus for given values of n, β and T, the values of ζ_0 and ζ_1 are independent of f_L which affects the critical slenderness ratio only in so far as f_L appears explicitly in equations (56) and (57). It is then easy to show from (49), (56) and (57) that (*a*) l/r_y decreases as f_L increases and (*b*) the product $f_L l/r_y$ increases as f_L increases. Hence, for given n, β and T it is a safe approximation (i) to use the values of l/r_y quoted for $f_L = 15{\cdot}25$ tons/sq.in. as applying for all smaller values of f_L and (ii) when $f_L > 15{\cdot}25$ tons/sq.in., to correct the tabulated values of l/r_y on the assumption that the product $f_L l/r_y$ is constant.

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Appendix A. Evaluation of $H_1(\zeta)$ and $H_2(\zeta, \beta)$

A. 1. *Exact solution for $H_1(\zeta)$ for a strut with ends simply supported about the minor axis*

We consider a straight pin-ended elastic strut having a non-uniform flexural rigidity B_2 given by (26), and subjected to an axial load P only. The factor $H_1(\zeta)$ is then defined by the statement that $P = P_0 H_1(\zeta)$ is the lowest critical load for lateral buckling of the strut, where P_0 is given by (22) and is the lowest critical load for the case of uniform flexural rigidity B_y. It follows that $H_1 = 1$ when $\zeta = 1$.

The governing equation for buckling of the strut is

$$B_2(z)\,(\mathrm{d}^2u/\mathrm{d}z^2) + P_0 H_1(\zeta)\,u = 0 \tag{A 1}$$

with end conditions

$$u = 0, \quad z = 0, \tag{A 2}$$

$$u = 0, \quad z = l. \tag{A 3}$$

Hence, from equations (A 1) and (26), we obtain

$$\frac{\mathrm{d}^2u}{\mathrm{d}z_1^2} + b^2 u = 0 \quad (0 \leqslant z_1 \leqslant \zeta), \tag{A 4}$$

$$\frac{\mathrm{d}^2u}{\mathrm{d}z_1^2} + b^2\left(\frac{1-\zeta}{1-z_1}\right)u = 0 \quad (\zeta \leqslant z_1 \leqslant 1), \tag{A 5}$$

where, with use of (22),

$$\left.\begin{aligned} b^2 &= \pi^2 H_1(\zeta), \\ z_1 &= z/l. \end{aligned}\right\} \tag{A 6}$$

The solution of (A 4) which satisfies (A 2) is

$$u = A' \sin bz_1 \quad (0 \leqslant z_1 \leqslant \zeta), \tag{A 7}$$

while the solution of (A 5) can be expressed in terms of Bessel functions (Jahnke & Emde, *Tables of functions*, p. 147, Dover 1945) and we find

$$u = B'\surd(1-z_1)\; J_1[2b\surd\{(1-\zeta)\,(1-z_1)\}] \quad (\zeta \leqslant z_1 \leqslant 1), \tag{A 8}$$

as the solution which satisfies the end condition (A 3).

The remaining conditions are that the deflexion u and the slope $\mathrm{d}u/\mathrm{d}z_1$ are continuous at $z_1 = \zeta$, whence from (A 7) and (A 8) we find

$$\left.\begin{aligned} A' \sin b\zeta &= B'\surd(1-\zeta)\, J_1[2b(1-\zeta)], \\ A' \cos b\zeta &= -B'\surd(1-\zeta)\, J_0[2b(1-\zeta)], \end{aligned}\right\} \tag{A 9}$$

and the elimination of the ratio A'/B' gives

$$\tan b\zeta = -\frac{J_1[2b(1-\zeta)]}{J_0[2b(1-\zeta)]} \quad (0 \leqslant \zeta \leqslant 1), \tag{A 10}$$

as the eigenrelation governing the critical loads for the strut.

For a given ζ, the lowest critical load $P = P_0 H_1(\zeta) = P_0 b^2/\pi^2$ is given by finding the smallest non-zero root of (A 10) regarded as an equation in b for given ζ. For numerical evaluation, it is simpler to solve (A 10) indirectly by writing

$$b' = 2b(1-\zeta), \tag{A 11}$$

whence equation (A 10) can be used explicitly to tabulate $b\zeta$ as a function of b'. We can then tabulate the corresponding ζ and $H_1(\zeta)$ by using

$$\zeta = \frac{2b\zeta}{2b\zeta+b'}, \quad H_1(\zeta) = \left(\frac{2b\zeta+b'}{2\pi}\right)^2. \tag{A 12}$$

The relevant range of ζ is $1 \geqslant \zeta \geqslant 0$ and the corresponding range of b' is $0 \leqslant b' \leqslant 3{\cdot}8317$, where this last value is the first non-zero positive root of $J_1(b') = 0$. The direct results of calculations are shown in table 3 and the corresponding plot of $H_1(\zeta)$ versus ζ is shown in figure 5. From such a plot, values of H_1 were read off for use in the main calculations.

A. 2. *Approximate evaluation of $H_2(\zeta, \beta)$ for a strut with ends simply supported about the minor axis*

We consider a straight elastic strut of uniform torsional rigidity C_0 and non-uniform minor flexural rigidity B_2 defined by (26). The relevant end conditions are (1) and (2), and the strut is subjected to major-axis end moments, M at the top and βM at the bottom, but no axial load. The factor H_2 is then defined by the statement that $M = M_0\sqrt{H_2}$ is the lowest critical value of M for lateral instability of the strut; here, M_0 is given by (22) and is the lowest critical moment for the case $\zeta = 1$ of a strut of uniform minor flexural rigidity B_y. It follows that $H_2 = 1$ when $\zeta = 1$ for any value of β when the ends of the strut are simply supported about the minor axis.

The governing equations for lateral buckling are obtained by putting $P = 0$, $u_0 = 0$ and $C = C_0$ in equations (4) and (5) to give

$$B_2(z)\frac{d^2u}{dz^2} = -M\phi f(z), \tag{A 13}$$

$$C_0\frac{d^2\phi}{dz^2} = Mf(z)\frac{d^2u}{dz^2}, \tag{A 14}$$

where B_2 is defined by (26), and from (22) we can write

$$M^2 = M_0^2 H_2(\zeta, \beta) = \pi^2 F(\beta)\, H_2(\zeta, \beta)\, (B_y C_0/l^2). \tag{A 15}$$

We seek, for given β and ζ, the smallest value of H_2 for which equations (A 13) and (A 14), with end conditions (1) and (2), have solutions for u and ϕ which are not identically zero.

Since a simple exact solution of these equations in terms of known functions could not be found, an energy method was used to obtain approximate values of H_2. Now, in the absence of axial load, the energy equation is

$$\frac{1}{2}\int_0^l B_2\left(\frac{d^2u}{dz^2}\right)^2 dz + \frac{1}{2}\int_0^l C_0\left(\frac{d\phi}{dz}\right)^2 dz = -M\int_0^l f(z)\,\phi\,\frac{d^2u}{dz^2}\,dz, \tag{A 16}$$

which, for a given strut and given β and ζ, we can regard as giving the value of M for any assumed forms of u and ϕ which satisfy the end conditions (1) and (2). It can then be shown by use of the calculus of variations that the smallest value of M for varying forms of u and ϕ corresponds to forms which satisfy the exact equations (A 13) and (A 14). For the approximate solution we shall restrict the shape of ϕ to a definite assumed type, and the corresponding shape of u will then be taken as the solution of (A 13) and (1) when the assumed form of ϕ is inserted in (A 13). Again by use of the calculus of variations it can be shown that this choice of u gives the least value of M for the assumed shape of ϕ. It is not in fact necessary to integrate (A 13) to determine u, since equation (A 16) can be expressed solely in terms of ϕ by substituting in it the value of the lateral curvature given by (A 13). In this way, the energy equation becomes

$$\frac{1}{2}\int_0^l C_0\left(\frac{d\phi}{dz}\right)^2 dz = \frac{M^2}{2}\int_0^l \frac{\phi^2 f^2}{B_2} dz, \qquad (A\,17)$$

which, by use of (A 15), gives the relation

$$\int_0^1 \left(\frac{d\phi}{dz_1}\right)^2 dz_1 = \pi^2 F(\beta)\, H_2(\zeta,\beta) \int_0^1 \frac{B_y}{B_2}\phi^2 f^2\, dz_1 \qquad (A\,18)$$

as the basic equation for determining H_2 as a function of ζ and β.

The shape chosen for ϕ is

$$\phi = z_1(1-z_1^m), \qquad (A\,19)$$

where m is positive so that ϕ satisfies the end conditions (2). This single-curvature shape seems reasonable physically for $\beta \geqslant 0$, and the index m can be used as a parameter varying with β and ζ to allow for some change in the shape of ϕ with different combinations of β and ζ. The best estimate for H_2 for given β and ζ will be its lowest value for varying m. Here we note that on an exact solution for $\beta < 1$ and $\zeta < 1$, the larger moment at the top end $z_1 = 1$ and the lower flexural rigidity near this end will both be expected to give a shape for ϕ which has a maximum value, for varying z_1, in the upper half of the strut. If this also holds for the assumed shape (A 19), it is to be expected that the least H_2 will occur for $m > 1$ and the numerical calculations confirmed this expectation.

The substitution of (A 19), (26) and (6) in (A 18) gives an equation from which the quantity

$$\lambda = \pi^2 F(\beta)\, H_2(\zeta,\beta) \qquad (A\,20)$$

can be evaluated numerically for given values of β, and m. Now for integral values $m = 1, 2, 3, \ldots$, the integrations in (A 18) can be performed analytically in terms of polynomials in ζ and the resulting numerical evaluation of λ is relatively simple. Accordingly, for a given (β,ζ), the value of λ was calculated for sufficient positive integral values of m to bracket the minimum value of λ for varying m. This minimum value and the associated value of m were then obtained by interpolation of the values for integral m. These calculations were carried out for each of the combinations (β,ζ) covered by $\beta = 0(0{\cdot}2)1$ and $\zeta = 0(0{\cdot}2)0{\cdot}6, 0{\cdot}7, 0{\cdot}8, 1{\cdot}0$. The results for $100/\lambda$ are given in table 4 and are given in this form instead of a table of values of H_2, since calculation of the critical slenderness ratio from the main analysis involves H_2 only in the combination $\lambda = \pi^2 F H_2$.

Apart from small errors of less than 1% arising from the interpolations used in the calculations, the values in table 4 will be in error due to the approximate nature of the energy method. This error will as usual be in the direction of overestimating the value of λ which is proportional to the square of the critical moment. Since this is an unsafe error, it is desirable to check that the error is not large. The following indications of the magnitude of the error have been obtained:

(*a*) For $\zeta = 1$, the exact value is $H_2(\zeta) = 1$ for all β by definition; accordingly, the values for $\zeta = 1$ in table 4 were compared with the exact values $100/\pi^2 F(\beta)$, based on Horne's table for $1/\sqrt{F}$, and indicated an error of less than 2% for all values of β in the tabulated range.

(*b*) When $\beta = 1$, it is easy to show that u satisfies equations which are identical in form to those for the strut under axial load only, i.e. equations (A 4) and (A 5). Hence it can be deduced that $H_2(\zeta, 1) = H_1(\zeta)$ for all ζ in the range $0 \leqslant \zeta \leqslant 1$, and noting that $F(\beta) = 1$ when $\beta = 1$, the values of $100/\lambda$ in table 4 can be compared with the values of $100/\pi^2 H_1$ obtained from the earlier exact solution for H_1. This comparison indicates an error increasing from about 0·5% when $\zeta = 0$ to about 1·5% when $\zeta = 1$.

(*c*) Two spot checks for $\zeta = 0$ were carried out by using the exact equations (A 13), (A 14) and (26) to obtain a single equation for ϕ and then solving this equation by the method of solution in power series. These laborious but relatively accurate calculations showed that for both $\zeta = 0, \beta = 0$ and $\zeta = 0, \beta = 0{\cdot}5$, the approximate values of table 4 are in error by rather less than $\frac{1}{2}$%.

The above comparisons suggest that for the whole range of the calculations, the error in the approximate values of table 4 is greatest when $\zeta = 1$ and that it is then at most about 2 %. This accuracy is considered sufficient for use in the main analysis of the paper.

A. 3. *Evaluation of $H_1(\zeta)$ when the foot of the strut is clamped about the minor axis*

The basic equations for determining $H_1(\zeta)$ are

$$B_2(z)\, \mathrm{d}^2u/\mathrm{d}z^2 + P_0 H_1(\zeta)\, u = (l-z)\, X, \tag{A 21}$$

$$\left.\begin{aligned} u = \mathrm{d}u/\mathrm{d}z = 0, \quad z = 0, \\ u = 0, \quad z = l, \end{aligned}\right\} \tag{A 22}$$

where B_2 is defined by (26). An exact solution involving Bessel functions can easily be obtained by a similar procedure to that used for the simply supported foot, and again writing $b^2 = \pi^2 H_1$, we find that

$$\frac{b \cos b\zeta - \sin b\zeta}{b \sin b\zeta + \cos b\zeta} = \frac{J_1[2b(1-\zeta)]}{J_0[2b(1-\zeta)]} \tag{23}$$

is the governing equation for buckling of the strut. This equation can be solved numerically in a similar way to that described for (A 10) and the final results are given in table 11. The value $H_1 = 2{\cdot}045$ for $\zeta = 1$ corresponds to the usual effect of clamping the foot of a uniform elastic strut. The effect of yielding is represented by the ratio $H_1(\zeta)/H_1(1)$ also shown in table 11. If these ratios are compared with the values of H_1 in figure 5 for the same ζ it is seen that the ratios in table 11 are smaller, thus indicating a relatively greater effect of yielding in reducing the critical load when the foot is clamped. Equivalently, it shows a decreasing beneficial effect from clamping the foot as yielding extends down the strut. This is to be expected physically, since buckling will depend increasingly on the less rigid upper portion of the strut as yielding extends down the strut, and the critical load will thus become less dependent on the foot conditions.

A. 4. *Evaluation of $H_2(\zeta, \beta)$ for a strut with foot clamped about the minor axis*

An approximate solution will be obtained by using an energy approach which differs only in detail from that used earlier for the strut with simply supported foot. The relevant

TABLE 11. VALUES OF $H_1(\zeta)$ FOR STRUT WITH ENDS CLAMPED ABOUT THE MINOR AXIS

ζ	$H_1(\zeta)$	$H_1(\zeta)/H_1(1)$
1	2·045	1
0·888	2·025	0·990
0·767	1·858	0·909
0·707	1·711	0·837
0·597	1·407	0·688
0·482	1·158	0·566
0·398	1·026	0·502
0·299	0·908	0·444
0·183	0·803	0·393
0·051	0·703	0·344
−0·026	0·651	0·318

energy equation is still (A 16) and a shape for ϕ will be assumed later, with the corresponding u then being defined as the solution of the exact equation

$$B_2(z)\,(\mathrm{d}^2u/\mathrm{d}z^2) = -M\phi f(z) + (l-z)\,X, \tag{A 24}$$

where ϕ is the assumed shape and the relevant end conditions are (A 22). Now, by substituting in (A 16) the value of $\mathrm{d}^2u/\mathrm{d}z^2$ given by (A 24), the energy equation becomes

$$\frac{1}{2}\int_0^l C_0\left(\frac{\mathrm{d}\phi}{\mathrm{d}z}\right)^2\mathrm{d}z = \frac{M^2}{2}\int_0^l \frac{\phi^2 f^2}{B_2}\,\mathrm{d}z - \frac{X^2}{2}\int_0^l \frac{(l-z)^2}{B_2}\,\mathrm{d}z, \tag{A 25}$$

which differs from the earlier equation (A 17) by involving the end shear reaction X. This must be chosen to satisfy the end restraints (A 22) and if we note that these conditions imply that

$$\int_0^l (l-z)\,\frac{\mathrm{d}^2u}{\mathrm{d}z^2}\,\mathrm{d}z = \left[(l-z)\,\frac{\mathrm{d}u}{\mathrm{d}z} + u\right]_0^l = 0 \tag{A 26}$$

and substitute therein the value of $\mathrm{d}^2u/\mathrm{d}z^2$ given by (A 24), we find that

$$X\int_0^l \frac{(l-z)^2}{B_2}\,\mathrm{d}z = M\int_0^l \frac{(l-z)\,\phi f}{B_2}\,\mathrm{d}z. \tag{A 27}$$

We now use this equation (A 27) to eliminate X from equation (A 25) and if we also use (A 15) to express M in terms of H_2, we obtain finally the equation

$$\frac{1}{\pi^2 F(\beta)\,H_2(\zeta,\beta)} = \frac{L_2}{L_1} - \frac{L_4^2}{L_1 L_3}, \tag{A 28}$$

where L_1, L_2, L_3, L_4 are defined by the integrals

$$\left.\begin{aligned} L_1 &= \int_0^1 \left(\frac{\mathrm{d}\phi}{\mathrm{d}z_1}\right)^2 \mathrm{d}z_1, \\ L_2 &= \int_0^1 \frac{B_y}{B_2}\,\phi^2 f^2\,\mathrm{d}z_1, \\ L_3 &= \int_0^1 \frac{B_y}{B_2}\,(1-z_1)^2\,\mathrm{d}z_1, \\ L_4 &= \int_0^1 \frac{B_y}{B_2}\,(1-z_1)\,\phi f\,\mathrm{d}z_1. \end{aligned}\right\} \tag{A 29}$$

Equations (A 28) and (A 29), where B_2 and f are defined by (26) and (6), respectively, enable an approximate estimate of H_2 to be obtained if we choose a form for ϕ satisfying the end conditions $\phi = 0$ at $z_1 = 0$ and $z_1 = 1$.

In the first place, the form (A 19) was used to evaluate H_2 and as a check on the accuracy of the approximate method, the results for $\beta = 1$ were compared with the exact values of H_1, since the relation $H_2(\zeta, 1) = H_1(\zeta)$ holds in the exact solution for the clamped foot in the same way as for the simply supported foot. It was found that when $\beta = 1$, the error ranged from about 4% to as much as 10%, the maximum error occurring for $\zeta = 1$. This error is much larger than the error of 2% noted earlier for the case of the foot simply supported about the minor axis. To explain this larger error, we note first that the shape $u(z)$ for the clamped foot will have as usual a point of inflexion where $\mathrm{d}^2u_1/\mathrm{d}z^2 = 0$. Secondly, equation (5) with $u = u_1$, $\phi = \phi_1$, and use of our assumption $C = C_0$, becomes

$$C_0 \frac{\mathrm{d}^2\phi_1}{\mathrm{d}z^2} = Mf(z) \frac{\mathrm{d}^2u_1}{\mathrm{d}z^2}, \tag{A 30}$$

so that the exact shape for ϕ_1 will also have a point of inflexion at the same section as $u_1(z)$. Thus the single-curvature shape (A 19) will not be as good a representation of the true shape $\phi_1(z)$ as it was for the strut simply supported about the minor axis. The error is likely to be largest when $\beta = 1$ since then $f = 1$ and we have

$$\frac{\mathrm{d}^2\phi_1}{\mathrm{d}z^2} = \frac{M}{C_0} \frac{\mathrm{d}^2u_1}{\mathrm{d}z^2} \tag{A 31}$$

and $u_1 = \phi_1 = 0$ at both $z = 0$ and $z = l$, so that for this case $\phi_1 = Mu_1/C_0$ and it follows that

$$\frac{\mathrm{d}\phi_1}{\mathrm{d}z} = \frac{M}{C_0}\frac{\mathrm{d}u_1}{\mathrm{d}z} = 0 \quad \text{at} \quad z = 0.$$

The shape of ϕ_1 is thus exactly similar to that of u_1 when $\beta = 1$ and may be expected to have a significant portion near the foot where the curvature is of opposite sign to that in the upper portion of the strut. On the other hand, when $\beta = 0$ we have $f(0) = 0$ and hence $\mathrm{d}^2\phi_1/\mathrm{d}z^2 = 0$ when $z = 0$. Thus for $\beta = 0$, the shape $\phi_1(z)$ has zero curvature both at the foot and at the section where $\mathrm{d}^2u_1/\mathrm{d}z^2 = 0$ and it is to be expected that the curvature between these sections will be small. Hence the single-curvature shape (A 19) may be expected to give appreciably greater accuracy for H_2 when $\beta = 0$ than when $\beta = 1$. Since no simple form for $\phi_1(z)$ could be devised to allow for the preceding variation of reverse curvature with β, it was decided to evaluate $100/\lambda = 100/\pi^2FH_2$ for both the single curvature shape (A 19) and the double-curvature shape

$$\phi = z_1^2 - z_1^{m+1}. \tag{A 32}$$

The more accurate value is then the larger of the two estimates of $100/\lambda$ for given (β, ζ) and this larger value is quoted in table 12. As expected, the shape (A 19) was more accurate for $\beta = 0$ and it was also found to be more accurate for $\beta = 0{\cdot}2$, $\zeta \leqslant 0{\cdot}8$ and the corresponding values of $100/\lambda$ in table 12 relate to using (A 19). The remaining values in table 12 are derived from the shape (A 32); in particular, for $\beta = 1$ this shape gives values of $100/\lambda$ which are in error by less than 3% in comparison with the exact value $100/\lambda = 100/\pi^2H_1$ for $\beta = 1$. For each shape, the maximum $100/\lambda$, giving least H_2, was found by using (A 28) and (A 29) to evaluate λ for integral values of m, and then interpolating to find the minimum λ for varying m.

TABLE 12. VALUES OF $100/\pi^2 FH_2$ FOR STRUT WITH ENDS CLAMPED ABOUT THE MINOR AXIS

$\beta \backslash \zeta$	0	0·2	0·4	0·6	0·7	0·8	1·0
0	10·8	8·78	6·81	4·77	3·75	2·90	2·37
0·2	11·5	9·30	7·26	5·11	4·07	3·22	2·73
0·4	12·2	9·91	7·75	5·49	4·42	3·62	3·17
0·6	13·1	10·7	8·38	6·00	4·87	4·09	3·66
0·8	14·1	11·5	9·09	6·55	5·38	4·61	4·21
1·0	15·2	12·4	9·85	7·16	5·95	5·19	4·82

APPENDIX B. ASSUMED SHAPE FOR $\psi(z_1)$ AND THE EVALUATION OF ζ_1

B. 1. *Strut with ends simply supported about the minor axis*

The function $\psi(z_1)$ represents the variation of $B_2\,\mathrm{d}^2u/\mathrm{d}z^2$ with $z_1 = z/l$, in the fundamental mode of buckling of a straight elastic strut with uniform torsional rigidity C_0 and non-uniform flexural rigidity B_2 defined by equation (26).

An exact determination of ψ for given values of β and ζ would require the solution of equation (8) with end conditions (9). If these equations are expressed in non-dimensional form, it is easily seen that the solution depends not only on β and ζ, but also on the relative magnitude of axial and moment loading as expressed by the ratio C_0P/M^2. However, in view of other approximations and assumptions in the analysis, great accuracy in the shape ψ is scarcely warranted and in particular we shall neglect the effect of C_0P/M^2 and base the shape on the case $P = 0$. This case has already been considered in connexion with the evaluation of H_2 (appendix A) and relatively little further calculation enables the relevant properties of ψ to be determined on the basis given below.

When $P = 0$, it follows from equation (10) that we can take the shape $\psi = \phi_1 f(z)$ where $f(z)$ is of the linear form given by (6) while ϕ_1 is the twisting rotation in the fundamental mode. We use the form (A 19) for ϕ_1 so that the assumed form for ψ is

$$\psi(z_1) = [z_1+\beta(1-z_1)]\, z_1(1-z_1^m), \tag{B 1}$$

where the relevant values of m are those giving the least value of H_2 for given β and ζ. Hence for these values of m it is easy from equation (B 1) to calculate the values of $\psi(\zeta)$ and also the values of ψ_m as functions of β and ζ; values of $\psi(\zeta)/\psi_m$ can then be obtained for use in evaluating $\mu(\zeta)$ from equation (48). Similarly, values of $\psi(\zeta)/m$ can be plotted as functions of ζ for given β, and then equation (45) can be conveniently solved graphically to determine ζ_1 which depends on n as well as β, since ζ_0 depends on both n and β as shown by equation (58). The results so obtained for ζ_1 are given in table 5.

Values of $\mu(\zeta_1)$ can either be obtained from equation (56) or by using

$$\mu(\zeta_1) = 0{\cdot}0015\pi^2\left(\frac{E}{f_L}\right)\frac{S(n)}{1-n}\left(\frac{m}{\psi_m}\right)\frac{1}{1-\beta} \tag{B 2}$$

which follows from (56) and (45), and is rather more accurate than the direct use of (56) in practice, especially when $\zeta_0-\zeta_1$ is small.

As noted earlier in appendix A, the numerical determination of H_1 and FH_2 is accurate to about 2 % or less and hence the derived values of $\alpha(\zeta)$, given by (57) for use in (49), are also relatively accurate. The corresponding values of $\mu(\zeta)$ and of ζ_1 are almost certainly less accurate since they are likely to be more sensitive to deviations of the assumed $\psi(z_1)$ from the exact solution for $\psi(z_1)$, owing in particular to the neglect of the effect of axial force on the form of ψ. However, values of $(l/r_y)_m$ will be little affected by such deviations since calculations show that this maximum value occurs for values of ζ where μ is small and hence this maximum slenderness ratio depends primarily on the function $\alpha(\zeta)$. Secondly, so far as the calculations of $(l/r_y)_{\zeta_1}$ are concerned, the use of an assumed form for ψ which is appropriate to case of zero axial load, i.e. $n = 0$, will introduce a safety margin for $n > 0$. Thus, increasing axial load will tend to make the true shape ϕ_1 of the fundamental mode more symmetrical, corresponding effectively to a smaller m than that for $P = 0$. Hence, use of values of m appropriate to $P = 0$ will tend to overestimate the negative contribution of $\psi'(1)$ to the inequality (42) derived from (25), and hence lead to a more stringent criterion than the true criterion for $P > 0$. Summing up, it is thought that the major source of error in ζ_1 and $\mu(\zeta_1)$ is likely to be on the safe side in underestimating $(l/r_y)_{\zeta_1}$, while on the other hand errors in μ lead only to much smaller errors in the value of $(l/r_y)_m$. In any case, it must be noted that μ is subject to an inevitable uncertainty arising from the assumed magnitude (30) for the initial curvature, and this uncertainty is probably larger than any other errors in μ.

B. 2. *Strut with ends clamped about the minor axis*

As earlier, we base the shape chosen for $\psi(z_1)$ on the case $P = 0$, but the relevant equation is now (A 24) with X given by (A 27). The shape used for ϕ was (A 19) for $\beta = 0$, 0·2 and (A 32) for $\beta = 0{\cdot}4$, 0·6 and 0·8. Thus, using the shape (A 19) we choose

$$\psi(z_1) = |\{z_1 - z_1^{m+1}\}\{z_1 + \beta(1-z_1)\} - (1-z_1)\,L_4/L_3| \quad (\beta = 0, 0{\cdot}2), \tag{B 3}$$

where L_3, L_4 can be obtained from (A 29) by using (26) and (A 19). Hence values of $\psi(\zeta)$ and ψ_m can be calculated for the value of m which gave the minimum λ for varying m and given β, ζ. It was found that $m > 1 > L_4/L_3$ in the range of the calculations and using (B 3), the criterion (42) becomes

$$\frac{\psi(\zeta)}{m - L_4/L_3} + \zeta \geqslant \zeta_0 \quad (\beta = 0, 0{\cdot}2), \tag{B 4}$$

and since the left-hand side of (B 4) was found to increase steadily with increasing ζ, the criterion may be written in the form

$$\left.\begin{aligned} &\zeta \geqslant \zeta_1 \\ &\frac{\psi(\zeta_1)}{m - L_4/L_3} + \zeta_1 = \zeta_0 \end{aligned}\right\} \quad (\beta = 0, 0{\cdot}2), \tag{B 5}$$

where the values of m and L_4/L_3 are those for $\zeta = \zeta_1$. The value of $\mu(\zeta_1)$ can then be obtained by using

$$\mu(\zeta_1) = 0{\cdot}0015\pi^2 \left(\frac{E}{f_L}\right) \frac{S(n)}{1-n} \left(\frac{m - L_4/L_3}{\psi_m}\right) \frac{1}{1-\beta}, \tag{B 6}$$

which follows from (56) and (B 5) and tends to be more accurate in practice than direct use of (56).

Similarly, for $\beta = 0{\cdot}4$, 0·6, 0·8, by using (A 32) in place of (A 19) we choose

$$\psi(z_1) = |\{z_1^2 - z_1^{m+1}\}\{z_1 + \beta(1-z_1)\} - (1-z_1)\,L_4/L_3|, \quad \beta = 0{\cdot}4, 0{\cdot}6, 0{\cdot}8, \tag{B 7}$$

and find

$$\left.\begin{aligned} \zeta \geqslant \zeta_1 \\ \frac{\psi(\zeta_1)}{m-1-L_4/L_3}+\zeta_1 = \zeta_0 \end{aligned}\right\} \quad (\beta = 0{\cdot}4, 0{\cdot}6, 0{\cdot}8), \tag{B 8}$$

$$\mu(\zeta_1) = 0{\cdot}0015\pi^2\left(\frac{E}{f_L}\right)\frac{S(n)}{1-n}\left(\frac{m-1-L_4/L_3}{\psi_m}\right)\frac{1}{1-\beta}, \tag{B 9}$$

where ψ_m, m and L_4/L_3 correspond to the value of m which gives the minimum value of H_2 for $\zeta = \zeta_1$ with use of the shape (A 32). The values of ζ_1 are given in table 13.

TABLE 13. VALUES OF ζ_1 FOR STRUT CLAMPED ABOUT THE MINOR AXIS

$\beta \backslash n$	0	0·1	0·2	0·4	0·6	0·8
0	0·791	0·730	0·685	0·642	0·633	0·624
0·2	0·756	0·683	0·628	0·576	0·563	0·550
0·4	0·700	0·600	0·525	0·450	0·440	0·420
0·6	0·610	0·465	0·310	0·140	0·100	0·065
0·8	0·307	—	—	—	—	—

APPENDIX C. A SAFE APPROXIMATION TO THE EIGENRELATION $F_1 = 0$

We consider a straight elastic strut with rigidities B_2, C, which are known functions of the distance z along the strut, independent of the loading. The strut is loaded by an axial load P and major-axis bending moments M and βM, and the end conditions are those of full torsional restraint and either simple support or clamping about the minor axis, corresponding to the two cases considered in the main analysis. The succeeding argument applies equally to either case, the essential points being that no work is done against the end restraints and that equations (2) and (5) hold in either case. For brevity, we write

$$\left.\begin{aligned} \int_0^l B_2\left(\frac{d^2u}{dz^2}\right)^2 dz &= I_1, \\ \int_0^l C\left(\frac{d\phi}{dz}\right)^2 dz &= I_2, \\ \int_0^l \left(\frac{du}{dz}\right)^2 dz &= I_3, \\ \int_0^l \phi f(z)\frac{d^2u}{dz^2}\,dz &= I_4, \end{aligned}\right\} \tag{C 1}$$

and the energy equation for the u and ϕ deformations is then

$$\tfrac{1}{2}I_1+\tfrac{1}{2}I_2 = \tfrac{1}{2}PI_3 - MI_4, \tag{C 2}$$

where the left-hand side represents the lateral bending and twisting energy stored in the strut, while the right-hand side gives the work done by the applied loads for small u and ϕ.

Now from equation (5) it follows that

$$MI_4 = \int_0^l \phi\frac{d}{dz}\left(C\frac{d\phi}{dz}\right)dz = -\int_0^l C\left(\frac{d\phi}{dz}\right)^2 dz = -I_2, \tag{C 3}$$

using integration by parts and the end conditions (2).

Hence from (C 2) and (C 3) it easily follows that

$$I_1 = PI_3 + M^2 I_4^2 / I_2 \tag{C 4}$$

which may be written in the form

$$\frac{P}{P_1} + \frac{M^2}{M_1^2} - 1 = 0, \tag{C 5}$$

where

$$\left.\begin{aligned} P_1 &= I_1/I_3, \\ M_1^2 &= I_1 I_2 / I_4^2. \end{aligned}\right\} \tag{C 6}$$

Thus, on an exact solution for the lowest mode where (u, ϕ) are equal to the pair of eigenfunctions (u_1, ϕ_1), the critical combination (P, M^2) for instability satisfies the relation (C 5). This may be regarded as the eigenrelation $F_1 = 0$, though it is not of course explicit, since (u_1, ϕ_1) are not known and their shapes will in general depend on the ratio Pl/M as well as z. We can however apply Rayleigh's principle to the relation (C 5) in order to obtain a safe approximation to the eigenrelation. For this purpose, we first regard M as fixed in value and the relation (C 5) as then giving the value of P for any chosen pair (u, ϕ) which satisfy the exact end conditions but not necessarily the exact equations of equilibrium. It is then straightforward, by use of the calculus of variations, to show that stationary values of P correspond to solutions of the exact equations of equilibrium so that, in particular, the least possible value of P will correspond to $(u, \phi) = (u_1, \phi_1)$ with (P, M^2) satisfying the eigenrelation $F_1 = 0$. Hence, for any other choice of admissible (u, ϕ), the value of P given by (C 5) for fixed M will be an overestimate of the true lowest critical value of P, which is a particular example of Rayleigh's principle. Let u_1, ϕ_1, P_1, M_1, refer to lowest critical mode for the general loading combination (P, M^2) and let u_{10}, ϕ_{10}, P_{10} similarly refer to the special case when $M = 0$. Now for $M = 0$, the relation (C 5) becomes simply $P = P_{10}$ for the exact solution (u_{10}, ϕ_{10}) whereas an approximate estimate $P = P_1$ would be obtained by using (u_1, ϕ_1) in place of (u_{10}, ϕ_{10}). Then by Rayleigh's principle

$$P_1 \geqslant P_{10}. \tag{C 7}$$

In an exactly similar manner, we can regard P as fixed and the relation (C 5) as then giving the value of M for any choice of admissible (u, ϕ) satisfying the end conditions. Rayleigh's principle then again follows in the form that the value of M given by (C 5) will be an overestimate of the true lowest critical value of M for the given P and by using the general (u_1, ϕ_1) as approximate shapes for the special case $P = 0$ we obtain the inequality,

$$M_1 \geqslant M_{10}, \tag{C 8}$$

where M_{10} is the true critical moment for lateral instability when $P = 0$.

It follows from (C 7) and (C 8) that the relation

$$\frac{P}{P_{10}} + \frac{M^2}{M_{10}^2} - 1 = 0 \tag{C 9}$$

is a safe approximation to the exact relation (C 5) since it will always give a lower value of P for given M, and a lower value of M for given P.

It should be noted that the relation (C 9) is exact for $P = 0$ and for $M = 0$ and it is linear in (P, M^2); hence it is represented graphically by the chord joining the end points of the curve $F_1 = 0$ in figure 1. The fact that the relation (C 9) is a safe approximation means that the curve $F_1 = 0$ lies wholly above the chord in figure 1 for $P > 0$, $M^2 > 0$. It may also be noted that an exact solution for the case of a uniform elastic strut has been obtained by Horne (1954) which indicates that in this case, the relation (C 9) is a close approximation to the exact relation $F_1 = 0$ for $\beta \geqslant 0$.

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